Deformation quantization:

Observables, states & representation theory

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Plan of the lecture:

I Overview: Deformation quantization

1) Motivation: Why quantization?
2) Canonical quantization, basic examples
3) General definitions and first results
4) Star products beyond quantization

II States and representations

1) The notion of positivity: ordered rings
2) *-Algebras over ordered rings
3) Positive functionals
4) The GNS construction

III Representation theory

1) Rieffel induction
2) Morita equivalence
3) Applications

1) Motivation: Why quantization?

- Quantum theory more fundamental than its classical counterpart.

- However: very difficult (impossible?) to find apriori quantum description in general.

- Thus: Quantize the classical picture!

  But how?

Correspondence principle:

* to any classical observable there corresponds a quantum observable
  Otherwise: „quantization" is hopeless...
* there is a **„classical limit"**
* Poisson-brackets correspond to commutators.

- How to make these ideas precise?
- Definition of „quantization", classical limit", „corresponds", ... ???

Separation of the quantization problem:

1) Generic features/questions.
Common to all „quantizations".

2) features/questions specific for examples
„what to do for the H_2 molecule?

Deformation quantization: focus on 1)
but also applications as 2) to obtain more specific answers/results.

In order to handle 1): first take a look at the „generic" classical situation.

Classical phase space M

$\hat{=}$ set of (pure) states

$\hat{=}$ set of initial conditions for the time development.

Functions on M have a Poisson bracket

$\Rightarrow$ M is a Poisson manifold, i.e. a smooth manifold with a smooth Poisson structure for $C^\infty(M)$

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

- bilinear and real: $\overline{\{f,g\}} = \{\bar{f}, \bar{g}\}$
- anti-symmetric: $\{f,g\} = -\{g,f\}$
- Leibnitz rule: $\{f, gh\} = \{f,g\}h + g\{f,h\}$
- Jacobi identity: $\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\}$

Observables $C^\infty(M)$ become a Poisson algebra in local coordinates $x^1, \dots, x^n$ of M

$$\{f, g\}(x) = \sum_{i,j} \alpha^{ij}(x) \frac{\partial f}{\partial x^i}(x) \frac{\partial g}{\partial x^j}(x)$$

with local functions $\alpha^{ij}(x) = -\alpha^{ji}(x)$

Jacobi identity $\iff$ $\forall\ i, j, k = 1, \dots, \dim M$

$$\sum_{\ell=1}^{\dim M} \left(\alpha^{i\ell} \frac{\partial \alpha^{jk}}{\partial x^\ell} + \alpha^{j\ell} \frac{\partial \alpha^{ki}}{\partial x^\ell} + \alpha^{k\ell} \frac{\partial \alpha^{ij}}{\partial x^\ell} \right) = 0$$

Poisson-tensor: $\alpha = \frac{1}{2} \alpha^{ij} \partial_i \wedge \partial_j$

Examples:

i) $M = \mathbb{R}^{2n}$, canonical Poisson-bracket

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right)$$

ii) $M = \mathfrak{g}^*$ where $\mathfrak{g}$ is a Lie algebra

$e_1, \dots, e_n$ Basis of $\mathfrak{g}$

$e^1, \dots, e^n$ dual basis of $\mathfrak{g}^*$

linear coordinates on $\mathfrak{g}^*$: $x = x_k e^k$

structure constants $[e_i, e_j] = c_{ij}^k e_k$

Then the Poisson bracket is defined by

$$\{f, g\}(x) = \sum_{i,j,k} x_k \, c_{ij}^k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

(Exercise: prove that this is a Poisson bracket...)

Def: A Poisson manifold (M, α) is called symplectic if in any local coordinate system the matrix $(\alpha^{ij}(x))$ is invertible at any point x.

In this case:

$$\omega = \tfrac{1}{2} \omega_{ij} \, dx^i \wedge dx^j, \quad (\omega_{ij}) = (\alpha^{ij})^{-1}$$

is a closed non-degenerate two-form „symplectic form"

and any such two-form gives a symplectic Poisson manifold.

Back to the examples:

i) The canonical Poisson bracket on $\mathbb{R}^{2n}$ is symplectic and

$$\omega = dq^i \wedge dp_i$$

ii) The Poisson bracket on $\mathfrak{g}^*$ is never symplectic since e.g.

$$\{f,g\}(0) \equiv 0$$

Theorem (Darboux):

If (M,ω) is symplectic then one can find local coordinates $q^1, \dots, q^n, p_1, \dots, p_n$ such that

$$\omega = dq^i \wedge dp_i$$

i.e. locally all symplectic manifolds look like $(\mathbb{R}^{2n}, \omega)$.

What kind of Poisson manifolds do occure in the physicists daily life?

* $(\mathbb{R}^{2n}, \omega)$

* if there are symmetries in the game then $\mathfrak{g}^*$ with the above Poisson structure always plays a role.

* $M = SO(3) \times \mathbb{R}^3$ phase space of rigid body

* $M = T^*Q$ if Q is the configuration space $\subseteq \mathbb{R}^{3N}$, specified by „constraints". There is a canonical Poisson bracket on T^*Q

* Physical systems with constraints / gauge degrees of freedom lead to a reduced phase space which can be quite complicated...

$\Rightarrow$ understanding quantization is still highly non-trivial task

2 Canonical quantization, basic examples

A) „Canonical" quantization on $\mathbb{R}^{2n}$

$$\rho: \quad \begin{array}{ccc} q & \longmapsto & Q = q\cdot \\ p & \longmapsto & P = -i\hbar \frac{\partial}{\partial q} \\ 1 & \longmapsto & \mathrm{id} \end{array}$$

+ ordering prescription for higher monomials

e.g.

standard ordering: $\rho_S(q^n p^m) = Q^n P^m$

Weyl ordering:

$\rho_W(q^n p^m) =$ totally symmetrized polynomial of $Q^n P^m$

$$\rho_S(q p^2) = \frac{1}{3}\left(QP^2 + QPQ + PQ^2\right)$$

explicitly:

$$\rho_S(f) = \sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{\hbar}{i}\right)^r \frac{\partial^r f}{\partial p^r}\bigg|_{p=0} \frac{\partial^r}{\partial q^r}$$

$$N = e^{\frac{i\hbar}{2}\Delta} \quad , \quad \Delta = \frac{\partial^2}{\partial q \partial p}$$

$$\rho_W(f) = \rho_S(Nf) = \sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{\hbar}{i}\right)^r \frac{\partial^r (Nf)}{\partial p^r}\bigg|_{p=0} \frac{\partial^r}{\partial q^r}$$

$\Rightarrow$ ρ_S, ρ_W can be extended to all smooth functions on $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ which are polynomial in the momentum coordinates p. Then

$$\boxed{\rho : \quad \mathrm{Pol}(T^*\mathbb{R}^n) \longrightarrow \mathrm{Diffops}(\mathbb{R}^n)}$$

is a linear bijection.

<u>Idea:</u> Pull-back the non-commutative product of $\mathrm{Diffops}(\mathbb{R}^n)$ via this „quantization map" ρ and obtain a new product for $\mathrm{Pol}(T^*\mathbb{R}^n)$.

Thus one defines the star products

$$f *_s g = \varrho_s^{-1}\left(\varrho_s(f)\,\varrho_s(g)\right)$$

$$f *_w g = \varrho_w^{-1}\left(\varrho_w(f)\,\varrho_w(g)\right)$$

explicitly:

$$f *_s g = \sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{\hbar}{i}\right)^r \frac{\partial^r f}{\partial p^r}\frac{\partial^r g}{\partial q^r}$$

$$f *_w g = N^{-1}\left(Nf *_s Ng\right)$$

with $N = e^{\frac{i\hbar}{2}\Delta}$ as before since $\varrho_w = \varrho_s \circ N$.

<u>Some properties:</u>

- $*$ is an associative product.
- $f * g = fg + o(\hbar)$
- $f * g - g * f = i\hbar\{f,g\} + o(\hbar^2)$
- $f * 1 = f = 1 * f$
- $f * g = \sum \hbar^r C_r(f,g)$ with bidiffops C_r

One has

$$\rho_S(f)^\dagger = \rho_S(N^2 \bar{f}) \neq \rho_S(\bar{f})$$

$\Rightarrow$ not a reasonable quantization as classical observables ($f = \bar{f}$) should correspond to quantum observables ($F = F^\dagger$)

But: $$\rho_W(f)^\dagger = \rho_W(\bar{f})$$

$\Rightarrow$ Weyl ordering is much better...

On the level of the deformed products this means

$$\overline{f \star_W g} = \bar{g} \star_W \bar{f}$$

but no nice relation for complex conjugation and $\star_S$

B) BCH star product on $\mathfrak{g}^*$ (after Simone Gutt)

$\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ polynomials on $\mathfrak{g}^*$
$\cong V^\bullet(\mathfrak{g})$ symmetric algebra over $\mathfrak{g}$ $\Big\}$ graded vector spaces

$\cong \mathcal{U}(\mathfrak{g})$ universal enveloping algebra of $\mathfrak{g}$
as filtered vector space.

(Poincaré - Birkhoff - Witt)

explicit isomorphism: total symmetrization

$$\sigma : x_1 \vee \dots \vee x_k \longmapsto \sum_{s \in S_k} x_{s(1)} \cdot \ldots \cdot x_{s(k)} \left(\tfrac{1}{k!}\right)$$

symmetric tensor product — product in $\mathcal{U}(\mathfrak{g})$

Then:

$$f *_{BCH} g := \sigma^{-1}\big(\sigma(f)\,\sigma(g)\big)$$

defines a non-commutative product for $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ such that it deforms the pointwise product in direction of the Poisson bracket in the same way as $*_W$ or $*_S$

More explicit formula $\Rightarrow$ extend '$*_{BCH}$' to exponential functions

$$e_x(\xi) := e^{x(\xi)} \qquad \begin{array}{l} x \in \mathfrak{g} \\ \xi \in \mathfrak{g}^* \end{array}$$

Then

$$e_X *_{BCH} e_Y = e_{\frac{1}{i t_1} BCH(i t_1 X,\, i t_1 Y)}$$

where $BCH(\cdot,\cdot)$ is the usual Baker-Campbell-Hausdorf series.

Problem: $BCH(x,y) = x + y + \frac{1}{2}[x,y] + \dots$

does not necessarily converge for all x, y

Thus: Consider the above formula as a formal power series in t_1

This is a general problem:

- Extension of $\star_W, \star_S, \star_{BCH}, \ldots$ to all smooth functions $C^\infty(M)$ not possible: Series in $\hbar$ generically diverge!!!

- Convergence only for nice subalgebras of $C^\infty(M)$

- However, on a generic phase space M there are no distinguished subalgebras of $C^\infty(M)$

Solution: First consider formal power series in $\hbar$.

Second, for a specific example, try to find nice subalgebras using specific extra information

3) General definitions and first results

Set-up: M: Poisson manifold $\hat{=}$ phase space

$C^\infty(M)$: smooth complex-valued functions on M $\Rightarrow$ Poisson algebra $\hat{=}$ classical observables

Definition: (BFFLS78)

A star product $\star$ for M is an associative $\mathbb{C}[[\lambda]]$-bilinear product for $C^\infty(M)[[\lambda]]$

$$f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f,g)$$

$\lambda \leftrightarrow \hbar$

such that

i) $f \star g = fg + \dots$

ii) $f \star g - g \star f = i\lambda \{f,g\} + \dots$

iii) $f \star 1 = f = 1 \star f$

iv) C_r is a bidifferential operator $\forall r$

v) it is called a Hermitian star product if $\overline{f \star g} = \overline{g} \star \overline{f}$

If $S = \mathrm{id} + \sum_{r=1}^{\infty} \lambda^r S_r$ is a formal series of differential operators on M and $\star$ is a star product then

$$f \star' g = S^{-1}(Sf \star Sg)$$

is again a star product (Exercise!)

<u>Example:</u> $\star_W$ and $\star_S$ are related this way by $N = e^{\frac{i\lambda}{2}\Delta}$.

Definition: Two star products related by such an S are called equivalent.

Question: Which physical properties of „quantization" do only depend on the equivalence class $[\star]$ and which do depend on the particular $\star$?

Some results:

Existence:

1983	DeWilde, Lecomte	symplectic case
1987	Fedosov	
1991	Omori, Maeda, Yoshioka	
1983	Gutt: linear Poisson structures	
⋮		
1997	Kontsevich: general Poisson case	
2000	Cattaneo, Felder give a TQFT interpretation of Kontsevich's formality theorem via the Poisson-Sigma models	

Classification:

1995	Nest, Tsygan	symplectic case
1997	Bertelson, Cahen, Gutt	
1997	Weinstein, Xu	
1997	Kontsevich: general Poisson case	

⇒ very strong existence & classification results make this approach to quantization perhaps the most successful one...

Remark: The mathematical framework is Gerstenhabers deformation theory of associative algebras.

Example: „Commuting derivations"

$\mathcal{A}$ associative algebra.

$D_i, E^i : \mathcal{A} \to \mathcal{A}$ pairwise commuting derivations

$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ undeformed product.

$\mu(a \otimes b) = ab$

Then

$$a \star b = \mu \circ e^{\lambda \sum_i D_i \otimes E^i} (a \otimes b)$$

is an associative deformed product for $\mathcal{A}[[\lambda]]$. (Exercise!)

4) Star products beyond quantization

"Metatheorem": Any associative deformation of a commutative algebra is „morally" a star product for some Poisson bracket.

Example: Quantum plane

Consider the vector fields $x\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$

$\Rightarrow$ commute!

$$f \star g = \mu \circ e^{i\lambda\, x\frac{\partial}{\partial x} \otimes y\frac{\partial}{\partial y}} f \otimes g$$

defines a star product for the Poisson bracket

$$\{f,g\} = xy\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)$$

„quadratic Poisson bracket"

Then

$$x \star y = \sum_{r=0}^{\infty} \frac{(i\lambda)^r}{r!} \, x y = e^{i\lambda} x y$$

$$y \star x = y x$$

$\Rightarrow$

$$x \star y = \underbrace{e^{i\lambda}}_{q} \, y \star x$$

Example: Noncommutative field theories

M Minkowski space

$B = \frac{1}{2} B_{ij} dx^i \wedge dx^j$ constant symplectic form

$\star$ Weyl product with respect to B

Then $(M, \star)$ „quantized space time"

$\Rightarrow$ field theories on $(M, \star)$?

idea: replace ordinary Lagrangeans by „non-commutative" ones

e.g.

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 + \alpha \phi^4 + \dots$$

is replaced by

$$\hat{\mathcal{L}} = \partial_\mu \phi \star \partial^\mu \phi - m^2 \phi \star \phi + \alpha \phi \star \phi \star \phi \star \phi + \dots$$

Advantage of star products:

- possible for <u>arbitrary</u> Poisson structures B
- gauge theories, etc ...
- even possible if the classical fields take their values in non-trivial vector bundle!

„Deformation quantization of vector bundles"

II States and representations

1) The notion of positivity: ordered rings

Up to now: a star product gives a model for the observable algebra build out of the classical observables.

What about the states?

- Vectors in a Hilbert space $\mathcal{H}$, but how should $(C^\infty(M)[[\lambda]], \star)$ act on a complex Hilbert space?
- Formal power series do not fit very well to operators on Hilbert spaces...

So is this already the point where one has to talk about convergence $\lambda \rightsquigarrow \hbar$?

..... NOT YET!

Guideline: C^*-algebras, here a state is an expectation value functional

$$\omega : \mathcal{A} \longrightarrow \mathbb{C} \quad , \text{ linear}$$

such that $\omega(A^*A) \geq 0$.

Example: $\mathcal{A}$ = bounded operators on Hilbert space $\mathcal{H}$

$\phi \in \mathcal{H}$, then $\omega : \mathcal{A} \to \mathbb{C}$

$$\omega(A) := \frac{\langle \phi \,|\, A\phi \rangle}{\langle \phi, \phi \rangle}$$

is a state for $\mathcal{A}$.

But also „mixed" states $\omega(A) = \operatorname{tr} \rho A$

Idea: Look for positive functionals of $C^\infty(M)[[\lambda]]$

First guess: $\omega : C^\infty(M)[[\lambda]] \longrightarrow \mathbb{C}$

$$\omega(\overline{f} * f) \geq 0$$

$\Rightarrow$ no interesting $\mathbb{C}$-linear functionals, convergence problem!

Better: $\omega : C^\infty(M)[[\lambda]] \longrightarrow \mathbb{C}[[\lambda]]$

now require $\mathbb{C}[[\lambda]]$-linearity.

But what should $\omega(\bar{f} \star f) \geq 0$ mean?

What is a positive formal power series?

Definition: $a = \sum_{r=r_0}^{\infty} \lambda^r a_r \in \mathbb{R}[[\lambda]]$ is called

positive if $a_{r_0} > 0$

This definition makes $\mathbb{R}[[\lambda]]$ an ordered ring

Definition: An associative, commutative, unital ring R is called ordered with positive elements $P \subset R$ if

$P \cdot P \subseteq P \qquad P + P \subseteq P$

$R = -P \,\dot{\cup}\, \{0\} \,\dot{\cup}\, P$

Examples: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}[[\lambda]]$,

If R is ordered $\Rightarrow$ $R[[\lambda]]$ is ordered

2) *-Algebras over ordered rings

In the following: Replace the numbers $\mathbb{R}$, $\mathbb{C}$ by "numbers" in an arbitrary ordered ring R and set

$$C = R \oplus iR \quad , i^2 = -1$$

$\Rightarrow$ For deformation quantization $R = \mathbb{R}[[\lambda]]$

Definition: A pre-Hilbert space $\mathcal{H}$ over C is a C-module with inner product $\langle \cdot , \cdot \rangle : \mathcal{H} \times \mathcal{H} \to C$ satisfying

i) $\langle \phi, z\psi + \omega\chi \rangle = z\langle \phi, \psi \rangle + \omega \langle \phi, \chi \rangle$

ii) $\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}$

iii) $\langle \phi, \phi \rangle > 0$ for $\phi \neq 0$

An operator $A : \mathcal{H} \to \mathcal{H}$ is called adjointable if there exists an operator A^* such that

$$\langle \phi, A\psi \rangle = \langle A^* \phi, \psi \rangle \qquad \forall \phi, \psi$$

Define (Hellinger-Toeplitz theorem !)

$$\mathcal{B}(\mathcal{H}) = \{ A \in \mathrm{End}(\mathcal{H}) \mid A^* \text{ exists} \}$$

and similar one defines $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$

Lemma: $A, B \in \mathcal{B}(\mathcal{H})$, $z, w \in \mathbb{C}$

i) A^* is unique, $A^* \in \mathcal{B}(\mathcal{H})$ and $A^{**} = A$.

ii) $zA + wB \in \mathcal{B}(\mathcal{H})$ and $(zA + wB)^* = \bar{z} A^* + \bar{w} B^*$

iii) $AB \in \mathcal{B}(\mathcal{H})$ and $(AB)^* = B^* A^*$

Analogous statements hold ~~for~~ $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$

Remark: Pre-Hilbert spaces over $\mathbb{C}$ form a category with objects $\mathcal{H}$ and morphisms $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.

Definition: An associative algebra $\mathcal{A}$ over $\mathbb{C}$ is called $*$-algebra if it is equipped with an involutive $\mathbb{C}$-antilinear antiautomorphism

$$*: \mathcal{A} \to \mathcal{A}$$

called the $*$-involution.
A $*$-homomorphism $\phi: \mathcal{A} \to \mathcal{B}$ is a algebra homomorphism with

$$\phi(A^*) = \phi(A)^*$$

Examples:

- Any C^*-algebra is a $*$-algebra over $\mathbb{C}$
- $(C^\infty(M)[[\lambda]], \star)$ with a Hermitian star product is a $*$-algebra over $\mathbb{C}[[\lambda]]$
- $\mathfrak{B}(\mathcal{H})$ for any pre-Hilbert space $\mathcal{H}$ over $\mathbb{C}$
- If $\mathcal{A}$ is a $*$-algebra over $\mathbb{C}$ then $M_n(\mathcal{A})$ is also a $*$-algebra over $\mathbb{C}$

Definition: A $*$-representation of $\mathcal{A}$ on a pre Hilbert space $\mathcal{H}$ is a $*$-morphism

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

Given two $*$-representations $(\mathcal{H}, \pi)$ and $(\mathcal{K}, \rho)$ of $\mathcal{A}$. A linear map $T : \mathcal{H} \to \mathcal{K}$ is called intertwiner if

$$T\,\pi(A) = \rho(A)\,T$$

Usually we require intertwiners to be isometric and/or adjointable.

Two $*$-representations are called equivalent if there exists a unitary intertwiner.

Definition: The representation theory of $\mathcal{A}$ is the category $*\text{-Rep}(\mathcal{A})$ of all $*$-representations with intertwiners as morphisms.

3) Positive functionals

Recall: We are looking for a notion of states in deformation quantization.

The star product algebras are particular $*$-algebras over $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ with $\mathbb{R}$ ordered.

Here we always assume the star product to be Hermitian $\overline{f \star g} = \overline{g} \star \overline{f}$.

Definition: Let $\mathcal{A}$ be a $*$-algebra over $\mathbb{C}$. A linear functional $\omega : \mathcal{A} \to \mathbb{C}$ is called positive if

$$\omega(A^* A) \geq 0$$

for all $A \in \mathcal{A}$.

We call $\omega(A)$ the expectation value of A in the state ω.

Remark: Sometimes we require $\omega(\mathbb{1}) = 1$

Examples:

- The δ-functional is not positive for the Weyl star product $\star_W$ since

$$\delta(\overline{H} \star_W H) = -\frac{\lambda^2}{4} < 0$$

where H is the Hamiltonian of the harmonic oscillator.

Physical interpretation:

Points in phase space are (in general) no longer states in quantum mechanics
$\Leftrightarrow$ uncertainty relations

- $f \in C_0^\infty(\mathbb{R}^{2n})[[\lambda]]$

$$\omega(f) = \int_{\mathbb{R}^n} f(q, p=0)\, d^n q$$

This turns out to be positive with respect to $\star_W$ (Exercise!)

„Schrödinger functional"

Question: How many positive functionals does $(C^\infty(M)[[\lambda]], \star)$ have?

<u>Classically</u>: The positive functionals of $C^\infty_0(M)$ are the compactly supported positive Borel measures

$$\omega(f) = \int_M f \, d\mu$$

Definition: A Hermitian deformation $\star$ of a *-algebra $\mathcal{A}$ over $\mathbb{C}$ is called a positive deformation if for any positive linear functional $\omega_0 : \mathcal{A} \to \mathbb{C}$ there exist 'quantum corrections' ω_r such that

$$\omega = \sum_{r=0}^{\infty} \lambda^r \omega_r : \mathcal{A}[[\lambda]] \longrightarrow \mathbb{C}[[\lambda]]$$

is positive with respect to $\star$.

Theorem (Bursztyn, W.)

Any Hermitian star product on a symplectic manifold is a positive deformation.

Remark:

- The example with the δ-functional shows that the 'quantum corrections' are indeed necessary (sometimes).

- Physically speaking, the theorem says that

  „any classical state is the classical limit of a quantum state."

4) The GNS construction

How can we get back to operators ... ?

Starting point: *-algebra $\mathcal{A}$ over $\mathbb{C}$
& positive functional
$\omega : \mathcal{A} \to \mathbb{C}$

Lemma: (Cauchy - Schwarz inequality)

$$\omega(B^*A) = \overline{\omega(A^*B)}$$

$$\omega(A^*B)\overline{\omega(A^*B)} \leq \omega(A^*A)\,\omega(B^*B) \qquad (1)$$

(Exercise: Prove this! Hint: ask some old Babylonian)

Consider now the following subset of $\mathcal{A}$

$$J_\omega := \{ A \in \mathcal{A} \mid \omega(A^*A) = 0 \}$$

$$\overset{(1)}{=} \{ A \in \mathcal{A} \mid \omega(B^*A) = 0 \quad \forall B \}$$

$\Rightarrow \mathcal{J}_\omega$ is a left ideal of $\mathcal{A}$

Thus the quotient

$$\mathcal{H}_\omega := \mathcal{A} / \mathcal{J}_\omega$$

becomes a left module for $\mathcal{A}$.

Notation: $\Psi_A \in \mathcal{H}_\omega$ denotes the equivalence class of $A \in \mathcal{A}$

Left module structure is then given by

$$\pi_\omega(A)\,\Psi_B = \Psi_{AB}$$

Furthermore, $\mathcal{H}_\omega$ is a pre-Hilbert space over $\mathbb{C}$ via

$$\langle \Psi_A, \Psi_B \rangle = \omega(A^* B)$$

Finally, π_ω is a $*$-representation

$$\langle \psi_A, \pi_\omega(B)\psi_C\rangle = \omega(A^* BC)$$

$$= \omega((B^* A)^* C) = \langle \pi_\omega(B^*)\psi_A, \psi_C\rangle$$

GNS representation of the positive functional ω

Examples:

i) $\mathcal{H} \ni \phi$ a vector with $\langle\phi,\phi\rangle = 1$,

$\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\omega(A) = \langle\phi, A\phi\rangle$

Then $\quad \mathcal{J}_\omega = \{A \mid A\phi = 0\}$.

$$\mathcal{H}_\omega = \mathcal{B}(\mathcal{H})/\mathcal{J}_\omega \cong \mathcal{H}$$

$$\psi_A \longmapsto A\phi$$

Thus we recover the usual action of $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$.

ii) The Schrödinger representation

$$\mathcal{A} = \left(C_0^\infty(T^*\mathbb{R}^n)[[\lambda]],\ *_W \right)$$

$$\omega(f) = \int_{\mathbb{R}^n} f(q, p=0)\, d^n q$$

Then the GNS representation of ω is canonically unitary equivalent to the Schrödinger representation in Weyl ordering ϱ_W on the pre-Hilbert space $C_0^\infty(\mathbb{R}^n)[[\lambda]]$ of „formal wave functions."

So one can go back to the „usual" quantum description by first specifying a star product and second a GNS representation w.r.t. a positive functional

Question: How many positive functionals does a *-algebra have?

There are examples of *-algebras without <u>any</u> positive functionals at all $\Rightarrow$ not usefull as observable algebras.

Definition: A *-algebra $\mathcal{A}$ over $\mathbb{C}$ has sufficiently many positive linear functionals if for any $0 \neq H = H^*$ $\in \mathcal{A}$ there exists a positive ω with $\omega(H) \neq 0$

Theorem: Let $\mathcal{A}$ have sufficiently many positive linear functionals. Then there exists a faithful (= injective) *-representation of $\mathcal{A}$.

Proof: Take the direct sum over all GNS representations.

Lemma: Let $(\mathcal{A}[[\lambda]], \star)$ be a Hermitian and positive deformation of $\mathcal{A}$. Then $(\mathcal{A}[[\lambda]], \star)$ has sufficiently many positive linear functionals if $\mathcal{A}$ has.

Since $C^\infty(M)$ certainly has sufficiently many positive linear functionals (take the δ-functionals) it follows that Hermitian star products also have suff. many positive functionals.

$\Rightarrow$ $(C^\infty(M)[[\lambda]], \star)$ has a faithful *-representation.

Remark: Such *-representations can also be obtained more explicitly.

More applications of the GNS construction:

- There are analogues of $\star_W$ for general cotangent bundles T^*Q, having a Schrödinger-like representation on formal wave functions $C^\infty(Q)[[\lambda]]$.

- The WKB expansion can be obtained in a GNS representation where the positive functional is a particular integration over $\mathrm{graph}(dS) \subseteq T^*Q$ where $S: Q \to \mathbb{R}$ is a solution to the Hamilton-Jacobi equation $H \circ dS = E$.

- There is also a characterization of thermodynamical states using the KMS condition.

It turns out that the KMS states for given β and H are unique and of the form

$$\omega(f) = \mathrm{tr}\left(\mathrm{Exp}(-\beta H) * f\right)$$

where Exp is the $*$-exponential, and tr the (unique) trace of the algebra $\left(C^\infty(M)[[\hbar]], *\right)$

The GNS representation turns out to be faithful with commutant being (anti-) isomorphic to the algebra itself. This gives a sort of baby-version of the Tomita-Takesaki theorem

Conclusion:

- Star products have a 'rich' and physically relevant representation theory.

- The algebra of observables is the fundamental object in the sense that it determines its representations and states but not vice versa.

- This point of view allows to consider different representations of the same observable algebra (superselection rules ...).

$\Rightarrow$ Find tools to understand / describe the representation theory $^*\text{-Rep}(\mathcal{A})$