Bosonic string in $\mathcal{D}$-dimensional Minkowski space-time

Nambu-Goto action: $S = -\frac{1}{2\pi\alpha'}\int d^2\xi \sqrt{-g}$

$g_{mn} = \partial_m X^\mu \partial_n X_\mu$, $m = 0,1$; $\mu = 0, \dots \mathcal{D}-1$; $g = \det g_{mn}$

Equations of motion $\partial_m \left(\sqrt{-g}\, g^{mn} \partial_n X^\mu\right) = 0$

In general case eqs. are nonlinear.

Can be linearized by fixing reparametrization symmetry:

$\partial_{(+)}\partial_{(-)} X^\mu = 0$, $\xi^{(\pm)} = \xi^0 \pm \xi^1$

Gauge-fixing Virasoro constraints:

$\partial_{(+)} X^\mu \partial_{(+)} X_\mu = 0$ $\partial_{(-)} X_\mu \partial_{(-)} X^\mu = 0$

General solution of wave equations

$X^\mu(\xi^{(+)}, \xi^{(-)}) = X^\mu_L(\xi^{(+)}) + X^\mu_R(\xi^{(-)})$

Functions $X_{L,R}$ restricted by Virasoro constraints

So, $X_{L,R}$ describe both physical and superfluous degrees of fr.

Virasoro constraints should be solved to exclude superfluous degrees of freedom.

Geometrical approach

String worldsheet = surface embedded into Minkowski space-time

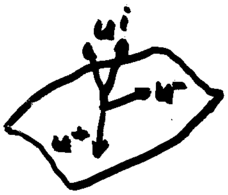
Embedding conditions : $dX^\mu = \frac{1}{2}\left(e^+(d)\, u^{-\mu} + e^-(d)\, u^{+\mu}\right)$

$e^\pm(d)$ — worldsheet vielbein 1-forms, related with metric

$g_{mn} = \frac{1}{2}\left(e^+_m e^-_n + e^-_m e^+_n\right)$ $e^\pm(d) = e^\pm_m d\xi^m$

u^μ_a — Lorentz harmonics, $a = 0, \dots \mathcal{D}-1$,

$u^\mu_a \in$ coset space $\frac{SO(1,\mathcal{D}-1)}{SO(1,1)\times SO(\mathcal{D}-2)}$



u^μ_a form local frame ; $u^\mu_a u_{\mu b} = \eta_{ab}$

u^+, u^- are tangent to worldsheet

u^i, $i = 1, \dots \mathcal{D}-2$, are orthogonal to worldsheet

Form of worldsheet is determined by equations, wich describe parallel transport of the basis along the world surface:

$du^a_\mu = \Omega^a{}_b u^b_\mu$, $\Omega^a{}_b \equiv u^a_\mu d u^\mu_b$ — Cartan 1-forms

Minimality condition: $d\left(e^+(d)\, u^{-\mu} - e^-(d)\, u^{+\mu}\right) = 0$

Main curvature of string worldsheet vanishes.

Set of embedding and minimality conditions for components of Cartan and vielbein forms is equivalent to bosonic string equations of motion for $X^\mu(\xi^{(+)}, \xi^{(-)})$.

All the equations can be reformulated in terms of Cartan and vielbein forms.

It is possible to fix reparametrization symmetry and solve Virasoro constraints:

$$\partial_{(+)} X_L^{\mu}(\xi^{(+)}) = \frac{1}{2} e^{-W-L} M^{+}_{(+)}(\xi^{(+)})\, u^{-\mu}$$

$$\partial_{(-)} X_R^{\mu}(\xi^{(-)}) = \frac{1}{2} e^{-W+L} M^{-}_{(-)}(\xi^{(-)})\, u^{+\mu}$$

where $W(\xi^{(+)},\xi^{(-)})$, $L(\xi^{(+)},\xi^{(-)})$ are scalar fields, parametrizing Cartan forms Ω^{+-}, $\Omega^{\pm i}$, Ω^{ij}

$M^{\dots}_{\dots}$ are chiral fields reflecting residual symmetries of the system.

After that most of the equations can be solved algebraically. Finally one get the set of equations:

$$\partial_{(+)} \partial_{(-)} W = \frac{1}{4} M^{-i}_{(+)} G^{ij} M^{+j}_{(-)} e^{2W}$$

$$\partial_{(-)} \left((\partial_{(+)} G)\, G^{T} \right)^{ij} = e^{2W} G^{[i|k} M^{+k}_{(-)} M^{-|j]}_{(+)}$$

where $G^{ij}(\xi^{(+)},\xi^{(-)})$ is orthogonal $SO(D-2)$ matrix field, parametrizing Cartan forms $\Omega^{\pm i}$, Ω^{ij}.

These equations are exactly solvable in a strong sense. The general solution has the form:

$$e^{-2W} = \frac{1}{2} r^{+}_{\mu} l^{-\mu}$$

$$G^{ij} = - l^{i}_{\mu} r^{j\mu} + \frac{r^{+}_{\mu} l^{i\mu} l^{-}_{\nu} r^{j\nu}}{r^{+}_{\rho} l^{-\rho}}$$

where

$$r^{a}_{\mu}(\xi^{(-)}) = (r^{\pm}_{\mu}, r^{i}_{\mu}) \in \frac{SO(1,D-1)}{SO(1,1) \times SO(D-2) \times K_{D-2}}$$

$$l^{a}_{\mu}(\xi^{(+)}) = (l^{\pm}_{\mu}, l^{i}_{\mu}) \in \frac{SO(1,D-1)}{SO(1,1) \times SO(D-2) \times K_{D-2}}$$

are two sets of left-moving and right-moving chiral Lorentz harmonics.

Wess - Zumino - Novikov - Witten model

Action: $$S = \frac{k}{16\pi} \int_M d\sigma\, d\tau\, \mathrm{Tr}\left(\partial_m \Omega\, \partial^m \Omega^{-1}\right) + \frac{k}{24\pi} \int_B d^3y\, \varepsilon^{abc}\, \mathrm{Tr}\left(\Omega^{-1}\partial_a \Omega\, \Omega^{-1}\partial_b \Omega\, \Omega^{-1}\partial_c \Omega\right)$$

$\xi^m = (\tau, \sigma)$ — coordinates on 2-dim. space-time M

y^a, $a = 1,2,3$ — coordinates on 3-dim. space-time B

$M = \partial B$

Ω - group valued field, $\Omega \in G$, G is a Lie group

Equations of motion:

$$\partial_{(-)}\left(\Omega\, \partial_{(+)} \Omega^{-1}\right) = 0 \qquad \partial_{(+)}\left(\Omega^{-1} \partial_{(-)} \Omega\right) = 0$$

$$\Omega\left(\xi^{(+)}, \xi^{(-)}\right), \qquad \xi^{(\pm)} = \tau \pm \sigma, \qquad \partial_{(\pm)} = \frac{\partial}{\partial \xi^{(\pm)}}$$

"Affine" symmetry:

$$\Omega\left(\xi^{(+)}, \xi^{(-)}\right) \longrightarrow L\left(\xi^{(+)}\right) \Omega\left(\xi^{(+)}, \xi^{(-)}\right) R\left(\xi^{(-)}\right), \qquad L, R \in G$$

Chiral currents: $M_{(+)} = -\Omega\, \partial_{(+)} \Omega^{-1}$, $\quad M_{(-)} = -\Omega^{-1} \partial_{(-)} \Omega$

$$\partial_{(-)} M_{(+)} = 0, \qquad \partial_{(+)} M_{(-)} = 0.$$

Chirality conditions are equivalent to the eqns. of motion.

Take $G = SO(1, D-1)$, Ω_{ab} is $D \times D$ matrix, $a,b = (+, i, -)$
$i = 1, \dots D-2$

Gauss decomposition: $\Omega = \Omega_B \Omega_K$

$$\Omega_B \in SO(1,1) \times SO(D-2) \times K_{D-2}$$
$$\Omega_K \in \frac{SO(1, D-1)}{SO(1,1) \times SO(D-2) \times K_{D-2}}$$

$$\Omega_B = \begin{pmatrix} 2e^{\omega} & 0 & 0 \\ -2e^{\omega} G^{ik} G^{+k} & -G^{ij} & 0 \\ 2e^{\omega} G^{+k} G^{+k} & 2G^{+j} & 2e^{-\omega} \end{pmatrix}$$

$$\Omega_K = \begin{pmatrix} 2 & 2G^{-j} & 2G^{-k}G^{-k} \\ 0 & -\delta^{ij} & -2G^{-i} \\ 0 & 0 & 2 \end{pmatrix}$$

e^{ω} — $SO(1,1)$ scale parameter

G^{ij} — $SO(D-2)$ orthogonal matrices

$G^{\pm i}$ — K_{D-2} Lorentz boosts parameters

First order equations:

$$\partial_{(+)}\omega = M^{-k}_{(+)} G^{+k} - \frac{1}{2} M^{+-}_{(+)}$$

$$\partial_{(-)}\omega = -M^{+k}_{(-)} G^{-k} + \frac{1}{2} M^{+-}_{(-)}$$

$$\partial_{(+)} G^{ij} = M^{ki}_{(+)} G^{kj} + M^{-i}_{(+)} G^{kj} G^{+k} - M^{-k}_{(+)} G^{kj} G^{+i}$$

$$\partial_{(-)} G^{ij} = M^{kj}_{(-)} G^{ik} - M^{+j}_{(-)} G^{ik} G^{-k} + M^{+k}_{(-)} G^{ik} G^{-j}$$

$$\partial_{(+)} G^{-i} = -\frac{1}{2} M^{-k}_{(+)} e^{-\omega} G^{ki}$$

$$\partial_{(-)} G^{+i} = \frac{1}{2} M^{+k}_{(-)} e^{-\omega} G^{ik}$$

$$\partial_{(+)} G^{+i} = \frac{1}{2} M^{+i}_{(+)} - M^{ik}_{(+)} G^{+k} + \frac{1}{2} M^{+-}_{(+)} G^{+i} + \frac{1}{2} M^{-i}_{(+)} G^{+k} G^{+k} - M^{-k}_{(+)} G^{+i} G^{+k}$$

$$\partial_{(-)} G^{-i} = -\frac{1}{2} M^{-i}_{(-)} - M^{ik}_{(-)} G^{-k} - \frac{1}{2} M^{+-}_{(-)} G^{-i} - \frac{1}{2} M^{+i}_{(-)} G^{-k} G^{-k} + M^{+k}_{(-)} G^{-i} G^{-k}$$

Second order equations for ω, G^{ij}

$$\partial_{(+)}\partial_{(-)}\omega = -\frac{1}{2} M^{-i}_{(+)} G^{ij} M^{+j}_{(-)} e^{-\omega}$$

$$\partial_{(-)}\left((\partial_{(+)} G) G^{T}\right)^{ij} = e^{-\omega} G^{[i|k} M^{+k}_{(-)} M^{-|j]}_{(+)}$$

are decoupled from the rest of equations for $G^{\pm i}$ and coincide exactly with the string equations when $\omega = -2W$. So, bosonic string nonlinear equations of motion are algebraically embedded into WZNW equations.

The connection between string theory and WZNW model is manifested on the general solution also.

WZNW model equations have the general solution, which could be expressed in terms of chiral harmonics

$$\Omega_a{}^b = l_a^{\mu} r_{\mu}^{b}$$

These equations recover the general solution for bosonic string equations and give the solution for $G^{\pm i}$ fields:

$$G^{+i} = \frac{r^{+}_{\mu} l^{i\mu}}{r^{+}_{\mu} l^{-\mu}} \qquad G^{-i} = \frac{l^{-}_{\mu} r^{i\mu}}{r^{+}_{\mu} l^{-\mu}}$$

and for the chiral currents:

$$M^{ab}_{(-)}(\xi^{(-)}) = r^{a\mu} \partial_{(-)} r^{b}_{\mu} \qquad M^{ab}_{(+)}(\xi^{(+)}) = l^{a\mu} \partial_{(+)} l^{b}_{\mu}$$