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Generalized Space-time Supersymmetries, Division Algebras and Octonionic M-theory

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TWO MAIN RESULTS

*)- OCTONIONIC M-ALGEBRA WHICH ADMITS 52 REAL BOSONIC GENERATORS INSTEAD OF 528 OF THE STANDARD M-ALGEBRA

*)- <u>NEW AND SURPRISING FEATURE</u> THE M5 SECTOR is NO LONGER INDEPENDENT, SUGGESTING THE EXISTENCE OF A DUAL PICTURE

$$M1 + M2 \underset{\text{versus}}{\Longleftrightarrow} M5 \text{ description}$$

WHY DIVISION ALGEBRAS (AND OCTONIONS)?

*)- AFTER KUGO-TOWNSEND - CONNECTION BETWEEN N-EXTENDED SUSIES AND DIVISION ALGEBRAS.

*) HIGHER-DIMENSIONAL FIELD THEORIES - The universal covering group of some Lorentz-group is isomorphic to division-algebra valued $sl(2)$ groups.

$$\widetilde{SO(2,1)} \sim Sl(2,\mathbb{R})$$

$$\widetilde{SO(3,1)} \sim Sl(2,\mathbb{C})$$

$$\widetilde{SO(5,1)} \sim Sl(2,\mathbb{H})$$

*) THE COMPACT, SIMPLY CONNECTED, PARALLELIZABLE SEVEN SPHERE CAN BE EXPRESSED THROUGH UNITARY OCTONIONS

$$S^7 = \{ x \in \mathbb{O} \mid x^* x = 1 \}$$

It fails to be a group-manifold due to the non-associativity of the octonionic algebra

RELATED WORKS (H.L. CARRION, M. ROJAS, F.T.)

*) [Superaffinization $\hat{\mathbb{O}}$ of the Malcev algebra of octonions
+ Sugawara $\hat{\mathbb{O}}$ $\rightsquigarrow$ $N=8$ S.C.A. (Non-associative)
of Englert et al.

Phys. Lett. A 2001

*) [Construction of $N=2,4,8$ Extended Super-KdVs.

*) [Classification and explicit construction of division algebra-valued Clifford algebras, spinors and their free dynamics
(in preparation)

*) [Extension to the non-associative case of the classification of the representation of 1D N-Extende Susy Quantum Mechanical Systems (A. Pashnev - F.T. in J. Math. Phys. [illegible])

One aspect of the problem: classifying the superalgebras going beyond the Haag-Łopuszański-Sohnius scheme.

Many works. E.g. Ferrara et al 2000/2001..

Real spinors Q_a

$$\{Q_a, Q_b\} = Z_{ab}$$

Z symmetric and abelian

Division-algebra valued spinors Q_a, Q_b^+

$$\{Q_a, Q_b\} = \{Q_a^+, Q_b^+\} = 0$$

$$\{Q_a, Q_b^+\} = Z_{ab}$$

$Z = Z^+$ Hermitian & Abelian

For M-algebra (Minkowski $D=11$, 32-component real spinors)

$$Z_{ab} = (C\Gamma_\mu)_{ab} P^\mu + (C\Gamma_{[\mu\nu]})_{ab} Z^{[\mu\nu]} + (C\Gamma_{[\mu_1\ldots\mu_5]})_{ab} Z^{[\mu_1\ldots\mu_5]}$$

DIVISION-ALGEBRA CONSTRAINT: PART OF THE GAME IS MATCHING SPINORIAL AND CLIFFORD Γ-MATRICES PROPERTIES.

	SPINORS		Γ's	
(1,0)	$\mathbb{R}$	1	$\mathbb{R}$	1
(1,1)	$\mathbb{R}$	1	$\mathbb{R}$	2
(1,2)	$\mathbb{R}$	2	$\mathbb{C}$	4
(1,3)	$\mathbb{C}$	4	$\mathbb{H}$	8
(1,4)	$\mathbb{H}$	8	$\mathbb{H}$	8
(1,5)	$\mathbb{H}$	8	$\mathbb{H}$	16
(1,6)	$\mathbb{H}$	16	$\mathbb{C}$	16
(1,7)	$\mathbb{C}$	16	$\mathbb{R}$	16
(1,8)	$\mathbb{R}$	16	$\mathbb{R}$	16
(1,9)	$\mathbb{R}$	16	$\mathbb{R}$	32
(1,10)	$\mathbb{R}$	32	$\mathbb{C}$	64

	Herm.	Antiher	Tot
32×32 $\mathbb{R}$-valued	528	496	1024
16×16 $\mathbb{C}$-valued	256	256	512
8×8 $\mathbb{H}$-valued	120	136	256
4×4 $\mathbb{O}$-valued	52	76	128

RECURSIVE CONSTRUCTION OF D+2 SPACETIME DIM. CLIFFORD ALGEBRAS FROM D-DIMENSIONAL ONES

$$\mathrm{I})-\left[\begin{array}{lll}\gamma_i & \mapsto & \Gamma_j \equiv \left\{ \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{1}_d \\ -\mathbb{1}_d & 0 \end{pmatrix}, \begin{pmatrix} \mathbb{1}_d & 0 \\ 0 & -\mathbb{1}_d \end{pmatrix} \right\} \\ (p,q) & \mapsto & (p+1, q+1) \end{array}\right.$$

$$\mathrm{II})-\left[\begin{array}{lll}\gamma_i & \mapsto & \Gamma_j \equiv \left\{ \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix}, \begin{pmatrix} \mathbb{1}_d & 0 \\ 0 & -\mathbb{1}_d \end{pmatrix} \right\} \\ (p,q) & \mapsto & (q+2, p) \end{array}\right.$$

REMARK 1: The two-dimensional real-valued Pauli matrices τ_A, τ_1, τ_2 realizing the $C(2,1)$ Clifford algebra are obtained applying I or II to 1, i.e. the one-dimensional realisation of $C(1,0)$

$$\tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

REMARK 2: All Clifford algebras are obtained by recursively applying I and II to <u>$C(1,0)$</u> and to the Clifford algebras of the series <u>$C(0, 3+4m)$</u>, m non-negative integer, which must be previously known.

TABLE WITH THE MAXIMAL CLIFFORD ALGEBRAS

1	2	4	8	16	32	64	128	256
$\underline{(1,0)} \Rightarrow$	$(2,1) \Rightarrow$	$(3,2) \Rightarrow$	$(4,3) \Rightarrow$	$(5,4) \Rightarrow$	$(6,5) \Rightarrow$	$(7,6) \Rightarrow$	$(8,7) \Rightarrow$	$(9,8) \Rightarrow$
		$\underline{(0,3)} \nearrow$	$(1,4) \rightarrow$	$(2,5) \rightarrow$	$(3,6) \rightarrow$	$(4,7) \rightarrow$	$(5,8) \rightarrow$	$(6,9) \rightarrow$
		$\searrow$	$(5,0) \rightarrow$	$(6,1) \rightarrow$	$(7,2) \rightarrow$	$(8,3) \rightarrow$	$(9,4) \rightarrow$	$(10,5) \rightarrow$
			$\underline{(0,7)} \nearrow$	$(1,8) \rightarrow$	$(2,9) \rightarrow$	$(3,10) \rightarrow$	$(4,11) \rightarrow$	$(5,12) \rightarrow$
			$\searrow$	$(9,0) \rightarrow$	$(10,1) \rightarrow$	$(11,2) \rightarrow$	$(12,3) \rightarrow$	$(13,4) \rightarrow$
						$\underline{(0,11)} \nearrow$	$(1,12) \rightarrow$	$(2,13) \rightarrow$
						$\searrow$	$(13,0) \rightarrow$	$(14,1) \rightarrow$
							$\underline{(0,15)} \nearrow$	$(1,16) \rightarrow$
							$\searrow$	$(17,0) \rightarrow$

REMARK 1: columns are labeled by the matrix size d of the maximal Clifford algebras

REMARK 2: The underlined Clifford algebras are the primitive maximal ones, the remaining maximal Clifford algebras are the maximal descendant Clifford algebras.

REMARK 3: Any non-maximal Clifford algebra is obtained from a given maximal one deleting a certain number of Γ-matrices.

EXPLICIT CONSTRUCTION OF THE PRIMITIVE MAXIMAL CLIFFORD ALGEBRAS OF THE QUATERNIONIC SERIES $C(0,3+8n)$ AND THE OCTONIONIC SERIES $C(0,7+8n)$.

$$C(0,3) \equiv \left.\begin{matrix} \tau_A \otimes \tau_1 \\ \tau_A \otimes \tau_2 \\ 1_2 \otimes \tau_A \end{matrix}\right\} \bar{\tau}_i \quad i=1,2,3$$

$$C(0,7) \equiv \left.\begin{matrix} \tau_A \otimes \tau_1 \otimes 1_2 \\ \tau_A \otimes \tau_2 \otimes 1_2 \\ 1_2 \otimes \tau_A \otimes \tau_1 \\ 1_2 \otimes \tau_A \otimes \tau_2 \\ \tau_1 \otimes 1_2 \otimes \tau_A \\ \tau_2 \otimes 1_2 \otimes \tau_A \\ \tau_A \otimes \tau_A \otimes \tau_A \end{matrix}\right\} \bar{\tau}_i \quad i=1,2,3,\dots,7$$

ASSOCIATIVE REALIZATION

$C_{\mathbb{O}}(0,7) \equiv e_i \Longleftarrow$ NON-ASSOCIATIVE REALIZATION

$$C(0,3+8n) = \left.\begin{matrix} \bar{\tau}_i \otimes \gamma_9 \otimes \dots & \otimes \gamma_9 \\ 1 \otimes \gamma_i \otimes 1 \otimes 1 \dots & \otimes 1 \\ 1 \otimes \gamma_9 \otimes \gamma_i \otimes 1 \otimes 1 \dots & \otimes 1 \\ 1 \otimes \gamma_9 \otimes \gamma_9 \otimes \gamma_i \otimes 1 \otimes 1 \dots & \otimes 1 \\ \dots & \\ 1 \otimes \gamma_9 \otimes \gamma_9 \otimes \dots & \gamma_9 \otimes \gamma_i \end{matrix}\right]$$

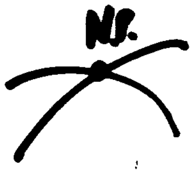
For associative Γ's $\quad \Sigma_{ij} = [\Gamma_i, \Gamma_j]$ is the generator of the Lorentz group.

Octonionic realization of $C(0,7)$ $\quad \Sigma_{ij} = [e_i, e_j] = C_{ijk} e_k$

$[e,e] \sim e \qquad 7 = \frac{1}{2} 7\cdot 6 - 14$

$\uparrow$ G_2-automorphisms group

It is a Malcev algebra: it describes the infinitesimal homogeneous transformations around the North Pole of S^7



$\bar{X} X = 1 \qquad x_0^2 + x_i^2 = 1$

$\delta x = a x \qquad a = a_0 e_0 + a_i e_i$

For $a_0 = 0$ the norm is preserved

Alternativity $\quad (a\cdot a) b = a \cdot (a b)$

$\Downarrow$

$J(a, b, [a,c]) = [J(a,b,c), a]$

(together with $[a,a] = 0$ defines a Malcev algebra).

Associative case $\binom{D}{k}$ antisymmetric k-tensor $\Gamma_{[\mu_1 \dots \mu_k]}$

Non-associative case: e.g $D = 11 \equiv 4 + 7$

real (4), imaginary octonionic (7)

the numbers are different.

In odd-dimensional space-times.

D \ k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
7	1	7	7	1	1	7	7	1						
9	1	9	22	22	10	10	22	22	9	1				
11	1	11	41	75	76	52	52	76	75	41	11	1		
13	1	13	64	168	267	279	232	232	279	267	168	64	13	1

$C\Gamma_{[\mu_1 \dots \mu_k]}$ Hermitian in $D = 11$ (M-theory)

standard case: $11 + 55 + 462 = 528$

$C\Gamma_1 \quad C\Gamma_{[\mu\nu]} \quad C\Gamma_{[\mu \dots \nu]}$

octonionic case $11 + 41 \;|\; 52 \qquad 52$

Careful treatment to deal with non-associative structures: you need a proper prescription. E.g.

$$[A_1 \cdot A_2 \cdot \ldots \cdot A_n] = \frac{1}{n!} \sum_{\text{perm.}} (-)^{\epsilon_{i_1 \ldots i_n}} (A_{i_1} \cdot A_{i_2} \cdot \ldots \cdot A_{i_n})$$

$$(A_1 \cdot A_2 \cdot \ldots \cdot A_n) = \frac{1}{2}(\ldots((A_1 A_2)A_3 \ldots)A_n) + \frac{1}{2}(A_1(A_2(\ldots A_n)\ldots))$$

is the correct one for the antisymmetrized product.

STANDARD CONSTRUCTION:

From ordinary super Poincaré algebra (defined in any dimension) the extension to conformal superalgebras can be realized only in $D=3,4,6$.

Superalgebras $U_\alpha U(4;n|\mathcal{F})$ for $\mathcal{F} \equiv \mathbb{R}, \mathbb{C}, \mathbb{H}$.

($U_\alpha U(n;m|\mathcal{F})$ is the algebra of the $\mathcal{F}$ valued graded transformations which preserve the bilinear form

$$\underline{q_i^{+} A_{ij} q_j} + \underline{\theta_k^{+} \theta_k},$$

θ_k $k=1,\dots,m$ are $\mathcal{F}$-valued Grassmann variables

$A_{ij} = -A_{ji}^{+}$ is antihermitian

q_i^{+} is the main conjugation in $\mathcal{F}$

For $D=3$ $U_\alpha U(4;n|\mathbb{R}) \equiv Osp(n;4|\mathbb{R})$ / $D=4,\dots$

In $D=11$ to introduce conformal superalgebras one needs to start from the M-algebra first.

$$\{Q_r, Q_s\} = Z_{rs} = (C\Gamma_\mu)_{rs} P^\mu + (C\Gamma_{[\mu\nu]})_{rs} Z^{[\mu\nu]} + (C\Gamma_{[\mu_1\dots\mu_5]})_{rs} Z^{[\mu_1\dots\mu_5]}$$

Introduce the conformal accelerations sectors $\tilde{Z}_{rs}$ by adding a second copy of the superalgebra $\{S_r, S_s\} = \tilde{Z}_{rs}$

$\tilde{Z}_{rs}$ symmetric 32×32 matrices $\qquad [Z_{rs}, Z_{kl}] = [\tilde{Z}_{rs}, \tilde{Z}_{kl}] = 0$

The crossed anticommutator $\{Q_r, S_s\} = L_{rs}$ is closed with the help of the Jacobi identities.

The 1024 generators $L_{\alpha s}$ form $GL(32;\mathbb{R})$

The algebra admits a 5-grading.

I_{-2}	I_{-1}	I_{0}	I_{1}	I_{2}
$\tilde{Z}_{rs}$	S_{α}	$L_{\alpha s}$	Q_{α}	Z_{rs}

F

B

$Osp(1|64)$ as conformal M-superalgebra.

Next: Extension of this construction to generalized Poincaré superalgebras in $D<11$.

Starting from $D=4$ we get $\mathcal{F}$-series for $D=4,5,7$

$\uparrow\ \uparrow\ \uparrow$

$\mathbb{R}\ \mathbb{C}\ \mathbb{H}$

$(3,1)$ $\{Q_{\alpha}, Q_{s}\} = Z_{rs}$ $\mathbb{R}^4 \times \mathbb{R}^4$

$(4,1)$ .... $\mathbb{C}^4 \times \mathbb{C}^4$

$(6,1)$ ... $\mathbb{H}^4 \times \mathbb{H}^4$

$D=4$	$4+6=10$
$D=5$	$5+11=16$
$D=7$	$7+21=28$

} Extended spacetime.

$D=4 \qquad \{Q_a, Q_b\} = C\Gamma_\mu P^\mu + C\Gamma_{[\mu\nu]} Z^{[\mu\nu]}$

$D=5 \qquad \{Q_a, Q_b^+\} = C\Gamma_\mu P^\mu + C\Gamma_{[\mu\nu]} Z^{[\mu\nu]} + CZ$

$D=7 \qquad \{Q_a, Q_b^+\} = C\Gamma_\mu P^\mu + C\Gamma_{[\mu\nu]} Z^{[\mu\nu]}$

To go to conformal superalgebra: construct a replica of the superalgebra (generators S_a, $\tilde{Z}_{ab}$) and introduce L_{ab} from the anticommutators $\{Q_a, S_b\} = L_{ab}$

$$L_{ab} \in GL(4|\mathcal{F})$$

$$\begin{array}{ccccc} I_{-2} & I_{-1} & I_0 & I_1 & I_2 \\ \Psi & \Psi & \Psi & \Psi & \Psi \\ \tilde{Z}_{ab} & S_a & L_{ab} & Q_a & Z_{ab} \end{array}$$

The bosonic sector is given by the conformal algebra $U_\alpha(8|\mathcal{F})$

The full conformal superalgebra is $UU_\alpha(8;1|\mathcal{F})$

For $D=4$ we get $OSp(1|8)$, ...

Comment about spin-algebras (Ferrara et al.)

For $D=3,4,6$ the spinorial covering of the conformal algebra $O(D,2)$ is described by $U_\alpha(4|\mathbb{F})$ (bosonic), i.e. a classical Lie group.

The situation is different for $D=5$, $D>6$. (this is why one needs to introduce extra-generators in super Poincaré)

Spin-algebra: Fundamental spinor representation of $O(n,m)$
$\mathbb{F}_{n,m}$ ($\mathbb{F}$-valued)

$Spin(n,m)$: group of $\mathbb{F}$-valued $N\times N$ endomorphisms of $\mathbb{F}_{n,m}$ containing the spinorial covering $\widetilde{O(n,m)}$

Minimal spin group: $Spin_{min}(n,m)$ containing the minimal number of generators.

Spinors $D=4$ $\mathbb{C}$ $\longrightarrow$ $\widetilde{Spin}_{min} = U_\alpha U(4;1|\mathbb{C})$

$D=5$ $\mathbb{H}$ $\longrightarrow$ $\widetilde{Spin}_{min} = U_\alpha U(4;1|\mathbb{H})$

$D=7$ $\mathbb{H}$ $\longrightarrow$ $\widetilde{Spin}_{min} = U_\alpha U(8;1|\mathbb{H})$

minimal conformal superalgebras.

For $D=7$ the two constructions coincide.
For $D=4,5$ they differ

FK

	minimal spin.	minimal Clifford
D=4	$\mathbb{C}$	$\mathbb{R}$
D=5	$\mathbb{H}$	$\mathbb{C}$
D=7	$\mathbb{H}$	$\mathbb{H}$

In D=7 the two proposals lead to the same (and minimal) conformal superalgebra because the division-algebra structure of spin and Clifford is the same.

For D=11 again we can use the same division-algebra this time is $\mathbb{O}$. In D=11 we get a minimal construction

M- superconformal algebra (conjectured)

The mixed bosonic generators $L_{ab} \in gl(4|\mathbb{O})$

Possibly this is a Malcev algebra?

Total # of bosonic generators $= 239$

Strong indication that octonionic M-theory should be relevant in defining M-theory on a compactified space

$$M(1,10) \longrightarrow M(1,3) \otimes S^7$$

A. PASHNEV and F.T. (JMP) CLASSIFICATION OF N-EXTENDED SUSIES IN (0+1)-DIMENSION (QUANTUM MECHANICAL SYSTEMS)

*) Made in several steps, using the specific properties of 0+1 supersymmetry.

E.g. the auxiliary field in a multiplet is a total derivative. In (0+1) it is a time derivative which can be algebraically solved.

$$\begin{array}{c} B\rightarrow \\ F\rightarrow \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{pmatrix} n_1\\ n_2\\ n_3\\ n_4\\ n_5\\ n_6\\ n_7\\ \vdots\\ n_k\\ n_{k+1}\end{pmatrix} \rightarrow \begin{pmatrix} n_1\\ n_2\\ n_3\\ \vdots\\ n_{k-1}+n_{k+1}\\ n_k\end{pmatrix} \Rightarrow \dots \begin{pmatrix} n=n_1+n_3+\dots\\ \\ n=n_2+n_4+\dots\end{pmatrix}$$

Shortest multiplets: $\begin{pmatrix} n\\ n\end{pmatrix}\begin{array}{l}\leftarrow \text{bosons}\\ \leftarrow \text{fermions}\end{array}$

The susy transformations are of the kind $\begin{pmatrix} 0 & \sigma\\ \tilde{\sigma}\cdot H & 0\end{pmatrix}$

The matrix $\begin{pmatrix} 0 & \sigma\\ \tilde{\sigma} & 0\end{pmatrix}$ is a special kind of a Clifford matrix which can be "promoted" to be a fermionic matrix of a superalgebra $\left(\begin{array}{c|c} & F\\ \hline F & \end{array}\right)$

$\exists$ a 1-to-1 CORRESPONDENCE BETWEEN EXTENDED SUSIES IN $D=1$ AND CLIFFORD ALGEBRAS OF WEYL TYPE

$$\boxed{\begin{array}{c} N = D \\ n = d \end{array}}$$

N = # of extended susies

n = # of states in an irrep. of extended susie.

D = dimensionality of the spacetime

d = dimensionality of the Γ's.

$\{Q_i, Q_j\} = \eta_{ij} H$

n	(p,q)
$(1+1)$	$(1,1)$
$(2+2)$	$(2,2)$
$(4+4)$	$(4,0)$ $(3,3)$ $(0,4)$
$(8+8)$	$(8,0)$ $(5,1)$ $(4,4)$ $(1,5)$ $(0,8)$
$(16+16)$	$(9,1)$ $(6,2)$ $(5,5)$ $(2,6)$ $(1,9)$
$(32+32)$	$(10,2)$ $(7,3)$ $(6,6)$ $(3,7)$ $(2,10)$

For $N=8$ $\exists$ a UNIQUE IRREP. ACTING ON $8+8$

explicitly constructed from the lifting II.

$$(0,7) \rightsquigarrow (9,0)$$

↳ drop 1 matrix, the remaining 8 are Weyl.

However $\exists$ $N=8$ (NON-ASSOCIATIVE) OBTAINED BY LIFTING THE OCTONIONIC REALIZATION OF $(0,7)$

$$C_{\mathbb{O}}(0,7) \rightsquigarrow C_{\mathbb{O}}(9,0) \Rightarrow N=8_{N.A.}$$

The two's $N=8$ are not equivalent.

Application to superextensions of KdV

*) GLOBAL SUPERSYMMETRY

*) Field dependent on (x,t). However susy-transformations only depend on x: We are back to the 1-Dim. SUSY case.

Investigation on possibility of $N=8$ extensions of KdV

Known fact: superconformal algebras for $N>4$ admit <u>no central extensions</u> (Leites, ...)

However the central extension is necessary to produce the term

$$u_t = u_{xxx} + \ldots \quad \text{in KdV}$$

<u>No $N=8$ KdV-extension for $N=8$ ASSOCIATIVE</u>

Non-associative $N=8$ S.C.A. (Englert et al. '88).

$$\Downarrow$$

global $N=8_{N.A.}$ transformation.

<u>$\exists\ N=8_{N.A.}$ KdV.</u> <u>It is unique and non-integrable.</u>

$$
\begin{cases}
\dot{T} = -T''' - 12T'T - 6Q_\alpha'' Q_\alpha + 4J_\alpha'' J_\alpha \\
\dot{Q} = -Q''' - 6T'Q - 6TQ' - 4Q_\alpha'' J_\alpha + 2Q_\alpha J_\alpha'' - 2Q_\alpha' J_\alpha' \\
\dot{Q}_\alpha = -Q_\alpha''' - 2QJ_\alpha'' - 6TQ_\alpha' - 6T'Q_\alpha + 2Q'J_\alpha' + 4Q''J_\alpha \\
\qquad - 2C_{\alpha\beta\gamma}\left(Q_\beta J_\gamma'' - Q_\beta' J_\gamma' - 2Q_\beta'' J_\gamma\right) \\
\dot{J}_\alpha = -J_\alpha''' - 4T'J_\alpha - 4TJ_\alpha' + 2QQ_\alpha' + 2Q'Q_\alpha - C_{\alpha\beta\gamma}\left(4J_\beta J_\gamma'' + 2Q_\beta Q_\gamma'\right)
\end{cases}
$$

However $\exists$ two $N=4$ parametric extensions which are potentially integrable for some values of the parameters. They could extend $N=4$ KdV preserving the integrability.

Remark: CONSISTENCY OF THE OCTONIONIC FORMULATION OF QUANTUM MECHANICS (ADLER), based on the Jordan formulation.

Günaydin-Piron-Ruegg Moufang Plane and Octonionic Quantum Mechanics C.M.P. (1978)

- Possible applications of the algebras here investigated.

Generalized top constrained [illegible] on S^7 instead of S^3.