HOW TO SOLVE NONCOMMUTATIVE GAUGE THEORY IN TWO DIMENSIO

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<u>LECTURE 1:</u> CLASSICAL ASPECTS

<u>LECTURE 2:</u> QUANTUM THEORY

<u>LECTURE 3:</u> PROPERTIES OF THE INSTANTON EXPANSION

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LECTURE 1 : CLASSICAL ASPECTS

1. The Noncommutative Torus

2. Yang-Mills Theory on the Noncommutative Torus (NCYM).

3. Classical Solutions of NCYM

~~PART I: NONCOMMUTATIVE GAUGE~~

~~THEORY IN TWO DIMENSIONS~~

THE NONCOMMUTATIVE TORUS

~~1: The Noncommutative Torus~~

A_θ = abstract, noncommutative, associative, unital $*$-algebra with 2 generators $\hat{Z}_1, \hat{Z}_2$ obeying the relations:

$$\hat{Z}_1 \hat{Z}_2 = e^{2\pi i\theta} \hat{Z}_2 \hat{Z}_1 \quad ; \quad \theta \in (0,1) \text{ (usually irrational)}$$

$$\hat{Z}_1^\dagger = \hat{Z}_1^{-1} \quad , \quad \hat{Z}_2^\dagger = \hat{Z}_2^{-1} .$$

"Smooth" completion consists of power series:

$$\hat{f} = \sum_{\vec{m}\in\mathbb{Z}^2} f_{\vec{m}}\, e^{\pi i \theta \overset{\text{symmetric ordering}}{m_1 m_2}}\, \hat{Z}_1^{m_1} \hat{Z}_2^{m_2}$$

$\hookrightarrow$ Schwartz sequences on $\mathbb{Z}^2$

Linear derivations: $\quad \hat{\partial} : A_\theta \longrightarrow A_\theta \otimes \mathfrak{L}_\Phi^*$ (linear).

$\hookrightarrow$ Heisenberg Lie algebra

$$[\hat{\partial}_1, \hat{\partial}_2] = i\Phi \quad , \quad [\hat{\partial}_i, \hat{Z}_j] = \frac{i}{R}\delta_{ij}\hat{Z}_i$$

$R \rightarrow$ "radius" of square torus

$$[\hat{\partial}_X, \hat{\partial}_Y] = \hat{\partial}_{[X,Y]} \quad , \quad X, Y \in \mathfrak{L}_\Phi .$$

Trace: $\quad \mathrm{Tr} : A_\theta \longrightarrow \mathbb{C} \quad ; \quad \mathrm{Tr}\, \hat{f}^\dagger \hat{f} \geq 0 \quad , \quad \forall \hat{f} \in A_\theta$

$$\mathrm{Tr}\,\hat{f} \equiv f_{\vec{0}} \qquad\qquad \mathrm{Tr}\,\hat{f}^\dagger = \overline{\mathrm{Tr}\,\hat{f}}$$

$$\mathrm{Tr}\,[\hat{\partial}_i, \hat{f}] = 0 .$$

Fields: View $A_\theta \equiv$ deformation of algebra of functions on $T^2 \to \mathbb{C}$: $C^\infty(T^2)$

$$\hat{f} \in A_\theta \quad \xleftrightarrow{1-1} \quad f(x) = \sum_{\vec{m} \in \mathbb{Z}^2} f_{\vec{m}}\, e^{i m_i x^i / R}$$

$$\hat{f}\hat{g} = \widehat{f * g}$$

$$(f*g)(x) = \sum_{\vec{m}, \vec{m}' \in \mathbb{Z}^2} f_{\vec{m}}\, g_{\vec{m}' - \vec{m}}\, e^{-\frac{i}{2}\theta\, \vec{m} \wedge \vec{m}'}\, e^{i m_i' x^i}$$

$\vec{m} \wedge \vec{m}' \to m_1 m_2' - m_2 m_1'$

$$= \frac{1}{\pi^2\theta^2} \iint d^2y\, d^2z\, f(x+y)\, g(x+z)\, e^{2i\, y \wedge z/\theta}$$

$$[\hat{\partial}_i, \hat{f}] = \widehat{\partial_i f}$$

$$\mathrm{Tr}\,\hat{f} = \frac{1}{A} \int d^2x\, f(x) \quad ; \quad A \equiv 4\pi^2 R^2 .$$

Cyclic Symmetry: $\int d^2x\, (f*g)(x) = \int d^2x\, f(x)\, g(x)$.

$$\equiv \mathrm{Tr}(\hat{f}\hat{g}).$$

Projective Modules: $\equiv$ "Vector bundles" over A_θ.

Serre-Swan $\Rightarrow$ $\left\{ \begin{array}{l} \text{Vector bundles} \\ \text{over } T^2 \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} \text{Finitely-generated} \\ \text{projective modules} \\ \text{over } C(T^2) \end{array} \right\}$

$$\begin{array}{ccc} E & & C(T^2, E) \\ \downarrow & \longleftrightarrow & \downarrow \\ T^2 & & C(T^2) \end{array}$$

$\mathcal{E}$ $\downarrow$ A_θ finitely-generated, projective (right) $\Longrightarrow$ $\mathcal{E} = p\, A_\theta^{\,n}$, $\quad A_\theta^{\,n} = \underbrace{A_\theta \oplus \cdots \oplus A_\theta}_{n \text{ times}}$

$p \in M_n(A_\theta)$, $\quad p^2 = p = p^\dagger$

$M_n(A_\theta) = A_\theta \otimes M_n$.

Connected components of infinite-dimensional manifold Gr_θ (5)

of Hermitian projectors of $\mathcal{A}_\theta$ parametrized by K-theory: (4)

$$K_0(\mathcal{A}_\theta) = \pi_1\Big(U_\infty(\mathcal{A}_\theta)\Big) = \mathbb{Z}\oplus\mathbb{Z} \quad \left(\begin{smallmatrix}\text{just like}\\ \text{for } T^2\end{smallmatrix}\right).$$

↳ equivalence classes of projectors modulo "stable isomorphism"

$Tr : \mathcal{A}_\theta \to \mathbb{C} \implies K_0(\mathcal{A}_\theta) \xrightarrow{\sim} \mathbb{Z} + \mathbb{Z}\theta \subset \mathbb{R}$ (isomorphism of ordered groups)

$$(p,q) \longrightarrow P_{p,q} \quad \text{with } Tr\otimes tr_m\, P_{p,q} = p - q\theta.$$

But: $\dim \mathcal{E} = Tr\otimes tr_m\, P = Tr\otimes tr_m\, PP^\dagger \geq 0$

$\Rightarrow$ stable projective modules are classified by positive cone of $K_0(\mathcal{A}_\theta)$.

$\Rightarrow (p,q)\in \mathbb{Z}\oplus\mathbb{Z} \longmapsto$ Heisenberg module $\mathcal{E}_{p,q}$ of >0

Murray-von Neumann dimension:

$$\dim \mathcal{E}_{p,q} = p - q\theta > 0.$$

All finitely-generated projective modules over $\mathcal{A}_\theta$ are either free modules ($\cong \mathcal{A}_\theta^N$) or Heisenberg modules.

NOTE: $q\in\mathbb{Z}$ is ~~Chern~~ character of "gauge bundle".

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<u>Connections</u>: $\hat{\nabla} : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}_\theta} \hat{L}_{\mathcal{I}}$ (homomorphism of vector spaces):

$$[\hat{\nabla}_i , \hat{Z}_j] = \frac{i}{R} \delta_{ij} \hat{Z}_j$$

$$\Rightarrow \hat{\nabla}_i = \hat{\partial}_i + \hat{A}_i \quad , \quad \hat{A}_i \in \mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}) = \mathcal{E}^* \otimes_{\mathcal{A}_\theta} \mathcal{E}$$

↳ really $\mathcal{F} \circ \hat{\partial}_i \circ \mathcal{F}$. "gauge fields"

↳ extended on $\mathcal{A}_\theta^n$ in obvious way.

Work only with <u>compatible</u> connections:

$$\hat{\nabla}_i \in CC(\mathcal{E}) \Rightarrow \langle \hat{\nabla}_i \hat{\xi} , \hat{\eta} \rangle_{A_\theta} + \langle \hat{\xi} , \hat{\nabla}_i \hat{\eta} \rangle_{\mathcal{A}_\theta} = [\hat{\partial}_i , \langle \hat{\xi} , \hat{\eta} \rangle_{\mathcal{A}_\theta}] .$$

$\langle \cdot , \cdot \rangle_{A_\theta} \equiv \mathcal{A}_\theta$ - valued inner product on $\mathcal{E}$, compatible with A_θ - module structure of $\mathcal{E}$.

$$\Rightarrow \text{Curvature } [\hat{\nabla}_1 , \hat{\nabla}_2] \in \mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E})$$

$$= [\hat{\partial}_1 , \hat{A}_2] - [\hat{\partial}_2 , \hat{A}_1] + [\hat{A}_1 , \hat{A}_2] + \Phi$$

$$\equiv \hat{F}_{\hat{A}} + \Phi$$

<u>Constant Curvature Connections</u>:

~~Every~~ Heisenberg module $\mathcal{E}_{p,q}$ admits a constant curvature connection $\hat{\nabla}^c \in C_{p,q} \equiv CC(\mathcal{E}_{p,q})$:

$$[\hat{\nabla}_1^c , \hat{\nabla}_2^c] = i f \quad ; \quad f \in \mathbb{R} \text{ constant.}$$

(Can set $\Phi = 0$, reinstate afterwards using Morita equivalence via the shift $f \longmapsto f + \Phi$).

Then $\mathcal{E}_{p,0} = L^2(T^2) \otimes \mathbb{C}^p$ (≡ free module of rank p).

$$\mathcal{E}_{p,q} \equiv L^2(\mathbb{R}) \otimes \mathbb{C}^q \quad , q \neq 0$$

Schrödinger rep. of $\mathcal{L}_\Phi$

$q \times q$ rep. of $SU(q)$ Weyl-'t Hooft algebra
$$\Gamma_1 \Gamma_2 = e^{2\pi i p/q}\, \Gamma_2 \Gamma_1$$

⇒ Solved explicitly by $SU(q)$ shift and clock matrices:

$$\Gamma_1 = (W_q^\dagger)^{2p} \quad , \quad \Gamma_2 = (V_q)^p$$

$$V_q = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ 1 & & & & 0 \end{pmatrix} \quad , \quad W_q = \begin{pmatrix} 1 & & & 0 \\ & e^{2\pi i/q} & & \\ & & \ddots & \\ 0 & & & e^{2\pi i(q-1)/q} \end{pmatrix}$$

Then represent generators of Noncommutative torus as:

$$\hat{Z}_i = e^{i f^{-1} \hat{\nabla}_i^c / R} \otimes \Gamma_i$$

$$\Rightarrow \theta = -\frac{2\pi}{Af} + \frac{p}{q}.$$

Weyl-'t Hooft algebra has a unique irrep. (up to $SU(q)$ equivalence) of dimension $\frac{q}{\gcd(p,q)}$

$$\Rightarrow \mathcal{E}_{p,q} = \underbrace{\mathcal{E}'_{p,q} \oplus \cdots \oplus \mathcal{E}'_{p,q}}_{N \text{ times}} \quad , \quad \mathcal{E}'_{p,q} = L^2(\mathbb{R}) \otimes \mathbb{C}^{q/\gcd(p,q)}.$$

⇒ defines rank of module $\mathcal{E}_{p,q}$:

$$N = \gcd(p,q).$$

$\Rightarrow \mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}_{p,q}) \cong \mathcal{A}_{\theta'}$, where

$$\theta' = \frac{n - s\theta}{p - q\theta} N \quad \equiv \underline{\text{dual}} \text{ noncommutativity parameter}$$

$$ps - qn = N \quad , \quad n, s \in \mathbb{Z} .$$

$$\Rightarrow \mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}_{p,q}) \cong \mathbb{M}_N(\mathcal{A}_{\theta'}) .$$

Also, Tr on $\mathcal{A}_\theta \rightsquigarrow \mathrm{Tr}_\mathcal{E} \equiv \mathrm{Tr} \otimes \mathrm{tr}_N$.

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2. YANG-MILLS THEORY ON THE NONCOMMUTATIVE TORUS (NCYM)

Define, for any $\hat\nabla \in CC(\mathcal{E})$, the Yang-Mills action functional:

$$S[\hat A] = S[\hat\nabla] = \frac{A}{2g^2} \mathrm{Tr}_\mathcal{E} [\hat\nabla_1 , \hat\nabla_2]^2$$

$$= \frac{1}{2g^2} \int d^2x \ \mathrm{tr}_N \left(F_A(x) + \Phi \right)^2$$

$$F_A = \partial_1 A_2 - \partial_2 A_1 + A_1 * A_2 - A_2 * A_1$$

$A_i(x) \equiv$ anti-Hermitian $U(N)$ gauge field.

$$* = \begin{pmatrix} \text{associative star-product} \\ \text{defined with } \theta' \end{pmatrix} \otimes \begin{pmatrix} \text{ordinary matrix} \\ \text{multiplication} \end{pmatrix} .$$

Gauge symmetry:

$$\hat\nabla_i \longmapsto \hat U \, \hat\nabla_i \, \hat U^\dagger$$

$$\hat U \in \mathrm{End}_{A_\theta}(\mathcal{E}) \ , \quad \hat U^\dagger \hat U = \hat U \hat U^\dagger = 1 \, .$$

Schwartz restriction on Fourier expansions

$$\Rightarrow \quad \hat U = 1 + \hat K \quad \in U^\infty(\mathcal{E}) \cong \overline{U(\infty)}$$

$\hat K \in$ algebra of <u>compact</u> endomorphisms of module $\mathcal{E}$.

$\equiv$ operator norm closure of algebra of finite-rank endomorphisms of $\mathcal{E}$:

$\hat\eta, \hat\eta' \in \mathcal{E} \Rightarrow$ define $|\hat\eta\rangle\langle\hat\eta'| \in \mathrm{End}_{A_\theta}(\mathcal{E})$ by

$$|\hat\eta\rangle\langle\hat\eta'| \, \hat\xi = \hat\eta \, \langle\hat\eta', \hat\xi\rangle_{A_\theta} \ , \quad \hat\xi \in \mathcal{E}.$$

$\Rightarrow$ A_θ-linear span forms self-adjoint 2-sided ideal in $\mathrm{End}_{A_\theta}(\mathcal{E})$

$\cong \mathbb{M}_\infty$ (since $\mathcal{E}$ is always separable).

Palais' Theorem $\Rightarrow \ \pi_k\big(U^\infty(\mathcal{E})\big) = \pi_k\big(U(\infty)\big) = \begin{cases} \mathbb{Z}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$

Defines group $\mathcal{G}(\mathcal{E})$ of gauge transformations.

<u>Infinitesimally:</u> $\quad \hat A_i \longmapsto \hat A_i + \delta_{\hat\lambda} \hat A_i$

$$\delta_{\hat\lambda} \hat A_i = [\hat\nabla_i, \hat\lambda] \ \Rightarrow [\delta_{\hat\lambda}, \delta_{\hat\lambda'}] \hat A_i = \delta_{[\hat\lambda, \hat\lambda']} \ldots$$

$\Rightarrow \mathcal{G}(\mathcal{E})$ acts on $\mathcal{C}(\mathcal{E})$

e.g.: $\quad \delta_\lambda A_i = \partial_i \lambda + \lambda * A_i - A_i * \lambda$.

Any $\mathcal{E}$ smoothly [illegible] matrix-valued field on T^2.

~~⊠~~ Noncommutative Gauge Transformations have a deep geometrical interpretation.

Define: $\hat{T}_{\vec{n}} = \frac{1}{\theta} e^{i n_1 n_2 \theta/2} \hat{Z}_1^{n_1} \hat{Z}_2^{n_2}$; $\vec{n} \in \mathbb{Z}^2$

$$\Rightarrow \left[\hat{T}_{\vec{n}} , \hat{T}_{\vec{m}} \right] = \frac{2i}{\theta} \sin\left(\theta \, \vec{n} \wedge \vec{m}\right) \hat{T}_{\vec{n}+\vec{m}}$$

In the limit $\theta \to 0$, this becomes the classical w_∞ algebra of area-preserving diffeomorphisms of T^2:

$$\left[\hat{T}^{(0)}_{\vec{n}} , \hat{T}^{(0)}_{\vec{m}} \right] = 2i \, \vec{n} \wedge \vec{m} \; \hat{T}^{(0)}_{\vec{n}+\vec{m}} .$$

$\Rightarrow$ Lie algebra of noncommutative gauge transformations is equivalent to an FFZ trigonometric deformation of the algebra $w_\infty(T^2)$ of area-preserving diffeomorphisms of T^2.

ie: NCYM is "almost" a topological field theory, because its gauge symmetry "almost" includes general covariance.
(not true in higher-D since then symplectic ≠ volume-preserving).

REMARK: $\theta = \frac{M}{N}$ (M, N co-prime) $\Rightarrow \hat{T}_{\vec{n}}$ algebra has a finite-D, N×N unitary representation ≡ FFZ trigonometric basis for $su(N)$

$\Rightarrow$ Limit $N \to \infty$ identifies gauge group of NCYM as a certain completion of $U(\infty)$.

Define a graded differential algebra:

$$\Omega(\mathcal{E}) = \bigoplus_{n \geq 0} \Omega^n(\mathcal{E}) \quad , \quad \Omega^n(\mathcal{E}) = \mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}) \otimes_{\mathcal{A}_\theta} \Lambda^n(\mathcal{L}_\Phi^*)$$

$\equiv$ left-invariant differential forms on $\exp(\mathcal{L}_\Phi)$ with coefficients in $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E})$.

eg: $\hat{F}_{\hat{A}} \in \Omega^2(\mathcal{E})$, with $\mathcal{G}(\mathcal{E})$ acting through

$$\delta_{\hat{\lambda}} \hat{F}_{\hat{A}} = [\hat{\lambda}, \hat{F}_{\hat{A}}] \,.$$

<u>Functional Differentiation at a point $\hat{A} \in CC(\mathcal{E})$:</u>

$$\frac{\delta}{\delta \hat{A}} f[\hat{a}] \equiv \frac{d}{dt} f[\hat{A} + t\hat{a}] \Big|_{t=0} \quad , \quad \hat{a} \in \Omega^1(\mathcal{E}).$$

- $CC(\mathcal{E})$ is an affine space over $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}) \otimes_{\mathcal{A}_\theta} \mathcal{L}_\Phi^*$, whose tangent space may be identified with $\Omega^1(\mathcal{E})$ above.

<u>Stationary Points of NCYM:</u>

$$\frac{\delta S[\hat{A}]}{\delta \hat{A}} = 0 \quad \Rightarrow \text{Noncommutative Yang-Mills equations of motion:}$$

$$\boxed{[\hat{\nabla}_i, [\hat{\nabla}_1, \hat{\nabla}_2]] = 0.}$$

ie: $\partial_i F_A + A_i * F_A - F_A * A_i = 0$.

We seek to characterize all such NCYM critical points.

Do this in a particular homotopy class of G_θ, characterized by $\mathcal{E} = \mathcal{E}_{p,q}$, $(p,q) \in \mathbb{Z}^2$ with $p - q\theta > 0$.

$\mathcal{E} = \mathcal{E}_{p,q}$ is characterized by $\hat\nabla^c \in \mathcal{C}_{p,q}$ of constant curvature:

$$\hat F_{\hat A^c} = \frac{2\pi}{A}\,\frac{q}{p - q\theta}.$$

These provide the absolute, global minimum value of NCYM on the module $\mathcal{E}_{p,q}$:

$$S[\hat\nabla^c + t\hat a] = \frac{A}{2g^2}\,\mathrm{Tr}_{\mathcal{E}_{p,q}}\left(\hat F_{\hat A + t\hat a} + \Phi\right)^2$$

$$= S[\hat\nabla^c] + \frac{At^2}{2g^2}\,\underbrace{\mathrm{Tr}_{\mathcal{E}_{p,q}}[\hat\nabla^c_i, \hat a]^2}_{\geq 0} + \mathcal{O}(t^4)$$

since $\mathrm{Tr}_{\mathcal{E}}[\hat\nabla_i, \hat\lambda] = 0$ generally.

$$\Rightarrow S[\hat\nabla^c + \hat a] \geq S[\hat\nabla^c] \quad \forall \hat a \in \Omega^1(\mathcal{E}_{p,q}).$$

We can also use constant curvature connections to construct <u>all</u> solutions to the classical equations of motion of NCYM:

Given any critical point $\hat\nabla = \hat\nabla^{cl}$, the curvature $\hat F_{\hat A^{cl}}$ corresponds to the central element of the Heisenberg Lie algebra generated by $\hat\nabla^{cl}_1$, $\hat\nabla^{cl}_2$, and $\hat F_{\hat A^{cl}}$.

$\hat F_{\hat A}$ generally acts on $\Omega(\mathcal{E}_{p,q})$ through the self-adjoint linear operators $\Xi_{\hat\nabla} : \Omega(\mathcal{E}_{p,q}) \to \Omega(\mathcal{E}_{p,q})$ defined by

$$\Xi_{\hat\nabla}(\hat\alpha) = [\hat F_{\hat A}, \hat\alpha] \;;\; \hat\alpha \in \Omega(\mathcal{E}_{p,q}),\; \hat\nabla \in \mathcal{C}_{p,q}.$$

Yang-Mills equations $\Rightarrow$ eigenvalues c_k c.. of $\Xi_{\hat{\nabla}}$ are constant near $\hat{\nabla} = \hat{\nabla}^{cl}$! $\Rightarrow$ eigenspace decomposition [12]

$$\Omega(\mathcal{E}_{p,q}) = \bigoplus_{k \geq 1} \Omega_{p_k, q_k} \;, \quad \text{where } \Xi_{\hat{\nabla}} \text{ acts on each } \Omega_{p_k, q_k}$$

as multiplication by a fixed scalar $c_k \in \mathbb{R}$.

$\Rightarrow$ Natural direct sum decomposition of module $\mathcal{E}_{p,q}$ into projective submodules $\mathcal{E}_{p_k, q_k}$!

$$\boxed{\mathcal{E}_{p,q} = \bigoplus_{k \geq 1} \mathcal{E}_{p_k, q_k}} \,.$$

Since $[\hat{F}_{\hat{A}^{cl}}, \hat{\nabla}_i^{cl}] = 0$, $\hat{\nabla}_i^{cl} : \mathcal{E}_{p_k, q_k} \longrightarrow \mathcal{E}_{p_k, q_k}$, and

$$\hat{\nabla}^c_{(k)} = \hat{\nabla}^{cl}\Big|_{\mathcal{E}_{p_k, q_k}} \quad \text{has constant curvature } \hat{F}_{\hat{A}^{cl}}\Big|_{\mathcal{E}_{p_k, q_k}} .$$

$\Rightarrow$ Define connection on $\mathcal{E}_{p,q}$ through $\hat{\nabla} = \bigoplus_{k \geq 1} \hat{\nabla}_{(k)}$, and use additivity of NCYM action!

$$S\Big[\bigoplus_{k \geq 1} \hat{\nabla}_{(k)}\Big] = \sum_{k \geq 1} S[\hat{\nabla}_{(k)}] \,.$$

Since each $S[\hat{\nabla}_{(k)}]$ is minimized by $\hat{\nabla}^c_{(k)}$, it follows that the Yang-Mills action has a critical point:

$$\hat{\nabla}^{cl} = \bigoplus_{k \geq 1} \hat{\nabla}^c_{(k)} \quad \text{on } \mathcal{E}_{p,q} = \bigoplus_{k \geq 1} \mathcal{E}_{p_k, q_k} \,.$$

<u>Every</u> classical solution of NCYM is of this form.

[illegible] important constraints on types of [illegible]

<u>decompositions</u> $\mathcal{E}_{p,q} = \bigoplus_{k\geq 1} \mathcal{E}_{p_k,q_k}$ [illegible]

(1) $$\dim \mathcal{E}_{p,q} = \sum_{k\geq 1} \dim \mathcal{E}_{p_k,q_k} \iff p - q\theta = \sum_{k\geq 1}(p_k - q_k\theta)$$

(2) $$q = \sum_{k\geq 1} q_k$$

$\iff$ K-theory charge conservation law $(p,q) = \sum_{k\geq 1}(p_k, q_k)$

distinguishes between "physical" NCYM and Yang-Mills theory defined on a particular projective module $\mathcal{E}_{p,q}$.

<u>Partitions:</u>

Any solution of the classical equations of motion of Yang-Mills on $\mathcal{E}_{p,q}$ $(\forall\theta)$ is completely characterized by a <u>partition</u>

$$(\vec{p}, \vec{q}) = \{(p_k, q_k)\}_{k\geq 1} \in \mathcal{P}_{p,q}(\theta)$$

$$\boxed{\begin{aligned} &p_k - q_k\theta > 0 \\ &\sum_{k\geq 1}(p_k - q_k\theta) = p - q\theta \\ &\sum_{k\geq 1} q_k = q \\ &0 < p_1 - q_1\theta \leq p_2 - q_2\theta \leq \dots \end{aligned}}$$

[illegible]

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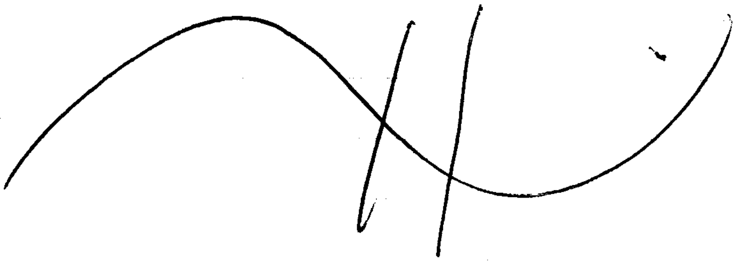
$$\boxed{\begin{aligned} S(\vec{p}, \vec{q}) &= S\Big[\bigoplus_{k \geq 1} \hat{\nabla}^{c}_{(k)} \Big] \\ &= \frac{2\pi^2}{g^2 A} \sum_{k \geq 1} (p_k - q_k \theta) \left(\frac{q_k}{p_k - q_k \theta} - \frac{q}{p - q\theta} \right)^2 \end{aligned}}$$

Natural boundary condition
$S(p, q; \theta) \equiv 0$,

with $\mathrm{Tr}_{\mathcal{E}_{p_k, q_k}} 1 = p_k - q_k \theta$.



LECTURE 2: {QUANTUM THEORY}

1. NCYM as a Hamiltonian System

2. Cohomology and the Quantum Partition Function

3. Ordinary ~~Yang-Mills~~ Quantum Gauge Theory Revisited

4. Morita Equivalence

Recall: $\mathcal{C}(\mathcal{E})$ is an affine space over vector space $End_{\mathcal{A}_\theta}(\mathcal{E}) \otimes_{\mathcal{A}_\theta} \mathcal{L}_\Phi^*$ of linear maps $\mathcal{L}_\Phi \to End_{\mathcal{A}_\theta}(\mathcal{E}) \Rightarrow$ tangent space to $\mathcal{C}(\mathcal{E})$ may be identified with ~~$\Omega^1(\mathcal{E}) \equiv End_{\mathcal{A}}$~~

$$\Omega^1(\mathcal{E}) \equiv End_{\mathcal{A}_\theta}(\mathcal{E}) \otimes_{\mathcal{A}_\theta} \Lambda^1(\mathcal{L}_\Phi^*) \ .$$

$\Rightarrow$ Define natural symplectic structure on $\mathcal{C}(\mathcal{E})$ by 2-form

$$\omega[\hat{a}, \hat{a}'] = Tr_{\mathcal{E}}\ \hat{a} \wedge \hat{a}' \quad ; \ \hat{a}, \hat{a}' \in \Omega^1(\mathcal{E}) .$$

This is clearly non-degenerate, and also independent of point $\hat{A} \in \mathcal{C}(\mathcal{E})$ at which it is evaluated. $\Rightarrow \omega$ is closed: $\dfrac{\delta\omega}{\delta\hat{A}} = 0$.

~~[illegible]~~ $\mathcal{G}(\mathcal{E})$ - Invariance:

$$\delta_{\hat{\lambda}}\hat{a}_i = [\hat{\lambda}, \hat{a}_i] \Rightarrow \omega[\delta_{\hat{\lambda}}\hat{a}, \delta_{\hat{\lambda}}\hat{a}'] = \omega[\hat{a}, \hat{a}'] .$$

Define a natural invariant, non-degenerate quadratic form on the Lie algebra of $\mathcal{G}(\mathcal{E})$ by:

$$(\hat{\lambda}, \hat{\lambda}') = Tr_{\mathcal{E}}\ \hat{\lambda}\hat{\lambda}' \quad ; \ \hat{\lambda}, \hat{\lambda}' \in End_{\mathcal{A}_\theta}(\mathcal{E}) .$$

$CC(\mathcal{E})$ contractible (since affine) + $\mathcal{G}(\mathcal{E})$ acts symplectically on $CC(\mathcal{E})$ $\Rightarrow$ $\exists$ moment map $\mu : CC(\mathcal{E}) \longrightarrow \left(\mathrm{End}_{A_\mathcal{E}}(\mathcal{E})\right)^*$ 3

naturally generating a system of Hamiltonians $H_{\hat\lambda} : CC(\mathcal{E}) \to \mathbb{R}$

$$\left(\mu[\hat A], \hat\lambda\right) = H_{\hat\lambda}[\hat A] .$$

<u>Construction of μ:</u>

Look at Hamiltonian flows $\Leftrightarrow$ $\mathcal{G}(\mathcal{E})$-invariance of ω:

$$\frac{\delta}{\delta \hat A} H_{\hat\lambda}[\hat a] = -\,\omega\left[\delta_{\hat\lambda}\hat A, \hat a\right] \quad ; \quad \hat a \in \Omega^1(\mathcal{E})$$

$$= -\,\mathrm{Tr}_\mathcal{E}\left[\hat\nabla, \hat\lambda\right] \wedge \hat a \qquad \text{by definition}$$

$$= \mathrm{Tr}_\mathcal{E}\left[\hat\nabla \,;\, \hat a\right]\hat\lambda \qquad \text{using Leibnitz} + \mathrm{Tr}_\mathcal{E}\left[\hat\nabla_i, \hat\lambda\right] = 0 .$$

$$= \frac{\delta}{\delta \hat A}\,\mathrm{Tr}_\mathcal{E}\left(\hat F_{\hat A} + \Phi\right)\hat\lambda$$

since $\hat F_{\hat A + t\hat a} = \hat F_{\hat A} + t\left[\hat\nabla \,;\, \hat a\right] + \mathcal{O}(t^2)$

near $\hat A \in CC(\mathcal{E})$.

$$\Rightarrow H_{\hat\lambda}[\hat A] = \left(\hat F_{\hat A} + \Phi , \hat\lambda\right)$$

$$\Rightarrow \boxed{\mu[\hat A] = \hat F_{\hat A} + \Phi .}$$

$\pi_1\left(\mathcal{J}^\infty(\mathcal{E})\right) = 0 \Rightarrow \hat\lambda \longmapsto H_{\hat\lambda}$ is a Lie algebra homomorphism on $\mathrm{End}_{A_\mathcal{E}}(\mathcal{E}) \longrightarrow \{$Poisson algebra on $CC(\mathcal{E})$ induced via ω above$\}$

We are now in a position to define the quantum theory, through the partition function: $\rightarrow$ c.f. $S[\hat{A}] \geqslant 0$

$$Z = \frac{1}{\mathrm{vol}\, \mathcal{G}(\mathcal{E})} \int_{\mathcal{C}(\mathcal{E})} \mathcal{D}\hat{A}\ e^{-S[\hat{A}]}$$

$\hookrightarrow$ determined from volume form on $\mathcal{G}(\mathcal{E})$ induced by metric $(\hat{\lambda}, \hat{\lambda})$ on $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E})$.

$$S[\hat{A}] = \frac{A}{2g^2}\left(\mu[\hat{A}], \mu[\hat{A}]\right).$$

We will ~~take $\mathcal{D}\hat{A}$ to be the phase~~ treat Z here as a phase space path integral corresponding to the above NCYM Hamiltonian system.

ie: $\mathcal{D}\hat{A}$ is taken to be the gauge-invariant Liouville measure on $\mathcal{C}(\mathcal{E})$.

To define it, let

$$d\hat{A} = \prod_{a,b=1}^{N} \prod_{x \in T^2} dA_1^{ab}(x)\, dA_2^{ab}(x)$$

be the usual, "ordinary" Feynman measure, and let $\hat{\psi}$ be the odd generators of the infinite-dimensional superspace:

$$\pi(\mathcal{E}) = \mathcal{C}(\mathcal{E}) \oplus \Pi\Omega^1(\mathcal{E}),$$

$\hookrightarrow$ [illegible] operator.

Let $d\hat{A}\, d\hat{\Psi}$ = corresponding (functional) Berezin measure. (20)

Then: $D\hat{A} = d\hat{A} \int_{\Pi\Omega^1(\mathcal{E})} d\hat{\Psi}\; e^{-i\omega[\hat{\Psi},\hat{\Psi}]}$ 5

(establish from usual Faddeev-Popov gauge fixing, coincides with that of commutative case).

Linearize action in $\mu[\hat{A}]$ via a functional Gaussian integration over an auxiliary field $\hat{\phi} \in \Omega^0(\mathcal{E})$:

$$Z = \frac{1}{\mathrm{vol}\, G(\mathcal{E})} \int_{\Omega^0(\mathcal{E})} d\hat{\phi}\; e^{-\frac{g^2}{2A}(\hat{\phi},\hat{\phi})}$$

$$\times \int_{\Pi(\mathcal{E})} d\hat{A}\, d\hat{\Psi}\; e^{-i\left(\omega[\hat{\Psi},\hat{\Psi}] - H_{\hat{\phi}}[\hat{A}]\right)}$$

$\hat{F}_{\hat{A}}$ is only place where noncommutativity is buried here

$\Rightarrow$ formally just like ordinary Yang-Mills

To analyse further the implications of this "hidden supersymmetry" through the basic multiplet $(\hat{A}_i, \hat{\Psi}, \hat{\phi})$, we study the cohomology of the operator:

$$Q_{\hat{\phi}} = \mathrm{Tr}_{\mathcal{E}}\left(\hat{\Psi}_i \frac{\delta}{\delta \hat{A}_i} + [\hat{\nabla}_i, \hat{\phi}] \frac{\delta}{\delta \hat{\Psi}_i} \right).$$

$$\Rightarrow \quad [Q_{\hat\phi}, \hat A_i] = \hat\psi_i ,$$

$$\{Q_{\hat\phi}, \hat\psi_i\} = [\hat\nabla_i, \hat\phi]$$

$$[Q_{\hat\phi}, \hat\phi] = 0$$

$$(Q_{\hat\phi})^2 = \delta_{\hat\phi} \qquad (\text{like BRST}).$$

In particular, the linearized action on $\pi(\varepsilon)$, for fixed $\hat\phi$, is closed under $Q_{\hat\phi}$:

$$Q_{\hat\phi}\left(\mathrm{Tr}_\varepsilon\, \hat\psi \wedge \hat\psi - H_{\hat\phi}[\hat A]\right) = 0 \qquad (\text{use Hamiltonian flows}).$$

Then, if $(Q_\phi)^2 \alpha = 0$; i.e.: α is gauge-invariant,

$$Z = Z_t = \frac{1}{\mathrm{vol}\, G(\varepsilon)} \int_{\Omega^0(\varepsilon)} d\hat\phi \; e^{-\frac{g^2}{2A}(\hat\phi, \hat\phi)}$$
$$\times \int_{\pi(\varepsilon)} d\hat A\, d\hat\psi \; e^{-i\left(\omega[\hat\psi, \hat\psi] - H_{\hat\phi}[\hat A] - t\, Q_{\hat\phi}\, \alpha[\hat A, \hat\psi]\right)} ,$$

is independent of $t \in \mathbb{R}$ [$\frac{d}{dt} Z_t = 0$ by Leibnitz for $Q_{\hat\phi}$ and integration by parts over $\pi(\varepsilon)$]

The $t \to \infty$ limit of this expression will receive contributions only from those fields for which $Q_{\hat\phi}\, \alpha[\hat A, \hat\psi] = 0$

$$\Rightarrow Z = \lim_{t\to\infty} Z_t$$

$$= \frac{1}{\mathrm{vol}\, G(\varepsilon)} \int_{\pi(\varepsilon)} d\hat{A}\, d\hat{\Psi}\; e^{-\mathrm{Tr}_\varepsilon\left(i\hat{\Psi}\wedge\hat{\Psi} + \frac{A}{2g^2}\mu[\hat{A}]^2\right)}$$

$$\times \lim_{t\to\infty} \exp\left(-\frac{A^3}{2g^2} t^2\, \mathrm{Tr}_\varepsilon \left[\hat{\nabla}^i \left[\hat{\nabla}_i, \mu[\hat{A}]\right]\right]^2\right)$$

$$\times \exp\left\{ i A t\, \mathrm{Tr}_\varepsilon \left(\mu[\hat{A}] [\hat{\Psi}_i, \hat{\Psi}^i] - [\hat{\nabla}_i, \hat{\Psi}^i][\hat{\nabla}\wedge\hat{\Psi}] \right)\right\}$$

$\Rightarrow$ Functional integral ~~[illegible]~~ becomes <u>localized</u> near solution of the equation:

$$\left[\hat{\nabla}^i, \left[\hat{\nabla}_i, \mu[\hat{A}]\right]\right] = 0 .$$

<u>Again</u> using Leibnitz $+ \mathrm{Tr}_\varepsilon[\hat{\nabla}_i, \hat{\lambda}] = 0$

$$\Rightarrow\; 0 = \mathrm{Tr}_\varepsilon\, \mu[\hat{A}] \left[\hat{\nabla}^i, [\hat{\nabla}_i, \mu[\hat{A}]]\right]^2$$

$$= -\mathrm{Tr}_\varepsilon \left[\hat{\nabla}_i, \mu[\hat{A}]\right]^2$$

$\Leftrightarrow [\hat{\nabla}_i, \mu[\hat{A}]] = 0 \rightsquigarrow$ classical equations of motion!!

So the partition function of NCYM receives contributions <u>only</u> from the space of solutions of the noncommutative Yang-Mills equations

Specifically: $Z_{p,q} = \sum_{\substack{\text{partitions} \\ (\vec{p},\vec{q})}} w(\vec{p},\vec{q})\, e^{[illegible]}$

$w(\vec{p},\vec{q})$ → quantum fluctuations about stationary points.

$\Rightarrow$ Expresses fact that ~~all~~ quantum noncommutative gauge theory in 2-dimensions is given exactly by sum over contributions from neighbourhoods of stationary points of the Yang-Mills action $S[\hat{A}]$.

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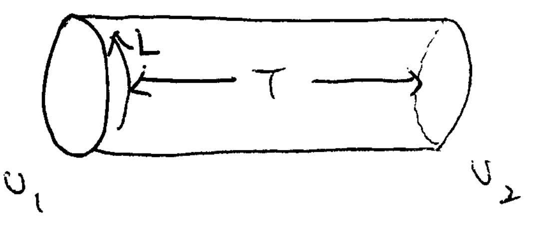
3. ORDINARY QUANTUM GAUGE ~~YANG-MILLS~~ THEORY REVISITED

Aim: Determine the fluctuation functions $w(\vec{p},\vec{q})$ above.



Physical Hilbert Space of U(p) Yang-Mills on a Cylinder:

Gauss' law $\Rightarrow$ $\Psi_{phys}[A] = \Psi[U]$

$$U = P\exp\left(\int_0^L dx\, A_1(x)\right).$$

gauge invariance $\Rightarrow$ Ψ depends only on conjugacy class

⇒ Hilbert space of physical states [illegible] space of L^2 - class functions on $G = U(p)$:

$$\mathcal{H}_{phys} = L^2\Big(U(p)\Big)^{Ad(U(p))},$$

$$\cong \bigoplus_R R \otimes \bar{R} \qquad \text{(Peter-Weyl)}.$$

↳ unitary irreps.

Representation basis provided by characters in unitary irreps.

⇒ states $|R\rangle$ have wavefunctions

$$\langle U | R \rangle = \chi_R(U) = \mathrm{Tr}_R(U),$$

<u>Hamiltonian</u>: $H = \frac{g^2}{2} L \,\mathrm{tr}\left(U \frac{\partial}{\partial U}\right)^2$ on $\Psi_{phys}[A]$

⇒ Hamiltonian is diagonalized in the ~~[illegible]~~ representation basis:

$$H = \frac{g^2}{2} L \, C_2(R)$$

↳ eigenvalue of quadratic Casimir in representation R.

<u>Cylinder Amplitude</u>:

$$Z(T, U_1, U_2) = \sum_R \chi_R(U_1)\chi_R(U_2^\dagger)\, e^{-\frac{g^2}{2} L T C_2(R)}$$

= heat kernel on the $U(p)$ group.

~~[illegible]~~ Orthogonality [illegible] group [illegible] elements

⇒ [illegible] $\int_{U(p)}$ [illegible] $= \delta_{R_1,R_2} \frac{[illegible]}{\dim R}$

↳ Haar measure on $U(p)$.

$\Rightarrow$ [illegible] amplitude:

$$Z = \int_{U(p)} [dU]\; Z(T, U, U) = \sum_R e^{-\frac{g^2 A}{2} C_2(R)}$$

To evaluate this ~~[illegible]~~, make sum over R's explicit $\Rightarrow$ each R is labelled by a set of p integers:

$$+\infty > n_1 > n_2 > \cdots > n_p > -\infty ,$$

$\hookrightarrow$ lengths of rows of Young Tableaux.

$$\Rightarrow C_2(R) = C_2(\vec{n}) \quad , \quad \vec{n} = (n_1, \dots, n_p)$$

$$= \frac{p}{12}(p^2 - 1) + \sum_{a=1}^{p} \left(n_a - \frac{p-1}{2} \right)^2$$

- $C_2(\vec{n})$ is symmetric under permutations of n_a's

$$\Rightarrow Z = \frac{1}{p!} \sum_{n_1 \neq \cdots \neq n_p} e^{-\frac{g^2 A}{2} C_2(\vec{n})} .$$

Extend sums over all $\vec{n} \in \mathbb{Z}^p$ by inserting:

$$\det_{1 \le a,b \le p} \left(\delta_{n_a, n_b} \right) = \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma) \prod_{a=1}^{p} \delta_{n_a, n_{\sigma(a)}} \; !$$

Permutation symmetry of $C_2(\vec{n}) \Rightarrow Z$ truncates to a sum over conjugacy classes of S_p

Classes $C[\vec{\nu}]$ labelled by partitions of p

$\nu_1 + 2\nu_2 + \cdots + p\,\nu_p = p$; ν_a = # elementary cycles of length a in $C[\vec{\nu}]$

with $\mathrm{sgn}\, C[\vec{\nu}] = (-1)^{p-|\vec{\nu}|}$, $|C[\vec{\nu}]| = \frac{p!}{\prod_{a=1}^{p} a^{\nu_a}\,\nu_a!}$.

$|\vec{\nu}| = \nu_1 + \nu_2 + \cdots + \nu_p$.

$$\sum_{n=-\infty}^{\infty} e^{-\pi g n^2 - 2\pi i b n} = \frac{1}{\sqrt{g}} \sum_{q=-\infty}^{\infty} e^{-\pi (q-b)^2/g} .$$

$$\Rightarrow Z = e^{-\frac{g^2 A}{24} p(p^2-1)} \sum_{\vec{v}:\, \sum_a a v_a = p} e^{i\pi\left(p + |\vec{v}| + (p-1)q\right)}$$

$$\times \prod_{a=1}^{p} \frac{\left(g^2 A a^3 / (2\pi^2)\right)^{-v_a/2}}{v_a!} \, e^{-S(\vec{v}, \vec{q})}$$

$$q \equiv q_1 + q_2 + \dots + q_{|\vec{v}|}$$

$$S(\vec{v}, \vec{q}) = \frac{2\pi^2}{g^2 A}\left(\sum_{k_1=1}^{v_1} \frac{q_{k_1}^2}{1} + \sum_{k_2=v_1+1}^{v_1+v_2} \frac{q_{k_2}^2}{2} + \sum_{k_3=v_1+v_2+1}^{v_1+v_2+v_3} \frac{q_{k_3}^2}{3} + \dots + \sum_{k_p = v_1+\dots+v_{p-1}+1}^{|\vec{v}|} \frac{q_{k_p}^2}{p} \right)$$

$\Rightarrow$ Agrees with expected sum over classical solutions (or <u>instantons</u>):

$K_0(C(T^2)) = \mathbb{Z} \oplus \mathbb{Z} \Rightarrow$ any $\mathcal{E} = \mathcal{E}_{p,q}$ is determined by $(p,q) \in \mathbb{Z}_+ \oplus \mathbb{Z}$, relatively prime, with $\dim \mathcal{E}_{p,q} = p > 0$ and constant curvature $\frac{q}{p}$.

Split bundle into subbundles:

$$E_{p,q} = \bigoplus_{k \geq 1} E_{p_k, q_k} \iff \mathcal{E}_{p,q} = \bigoplus_{k \geq 1} \mathcal{E}_{p_k, q_k}$$

$$\Rightarrow \operatorname{rank} E_{p,q} = p = \dim \mathcal{E}_{p,q}$$

$$= \sum_{k \geq 1} p_k = \nu_1 + 2\nu_2 + \cdots + p\nu_p ,$$

where $\nu_a \equiv$ # of sub-modules of dimension a (or sub-bundle of rank a).

NOTE: Magnetic charges $q_k \equiv$ dual to lengths of rows of Young tableaux of $U(p)$.

Now rewrite physical Yang-Mills theory ($\equiv$ sum of contributions from topologically distinct bundles over T^2) as ~~[illegible]~~ Yang-Mills theory defined on the isomorphism class $\mathcal{E}_{p,q}$ of projective modules:

$$Z = e^{-\frac{g^2 A}{24} p(p^2-1)} \sum_{q \in \mathbb{Z}} (-1)^{(p-1)q+p} Z_{p,q}$$

$$\boxed{Z_{p,q} = \sum_{\substack{\text{partitions} \\ (\vec{p},\vec{q})}} (-1)^{|\vec{\nu}|} \prod_{a=1}^{p} \frac{\left(g^2 A a^3 / 2\pi^2\right)^{-\nu_a/2}}{\nu_a !} e^{-S(\vec{p},\vec{q})}}$$

Gauge Morita equivalence $\Rightarrow$ 1-1 correspondence between projective modules over different noncommutative tori (ie: different θ's) associated with different topological numbers, and augmented with transformations of connections between the modules.

It is a symmetry of the Weyl action, which we will assume extends to the quantum level (good evidence for this).

Duality group $SO(2,2,\mathbb{Z})$ acts on K-theory ring ~~spin~~ $K_0(A) \oplus K_1(A)$ in a spinor representation.

$$SO(2,2,\mathbb{Z}) \cong SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$$

Kähler moduli of T^2 automorphisms of T^2

On $\mathcal{E} = \mathcal{E}_{p,q}$ it is defined as follows:

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} m & n \\ r & s \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \quad , \quad \begin{pmatrix} m & n \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z})$$

$\hookrightarrow$ doublet under $SL(2,\mathbb{Z})$

$$\theta' = \frac{m\theta + n}{r\theta + s}$$

$$\dim \mathcal{E}' = \frac{\dim \mathcal{E}}{|r\theta + s|}.$$

$$A' = (r\theta+s)^2 A$$

$$g'^2 = |r\theta+s|\, g^2$$

$$\Phi' = (r\theta+s)^2\,\Phi - \frac{2\pi}{A}\, r(r\theta+s)\,.$$

$$\begin{pmatrix}\text{const. curvature}\\ \text{connections}\end{pmatrix}' \overset{1-1}{\longleftrightarrow} \begin{pmatrix}\text{const. curvature}\\ \text{connections}\end{pmatrix}$$

$$\begin{pmatrix}\text{classical}\\ \text{solutions}\end{pmatrix}' \overset{1-1}{\longleftrightarrow} \begin{pmatrix}\text{classical}\\ \text{solutions}\end{pmatrix}.$$

~~Symmetry factors $\nu_a!$~~

Consider action now of Morita duality on quantum partition function of ordinary $\underset{}{\overset{\theta=0}{U(p)}}$ Yang-Mills theory on T^2 $\Rightarrow$ NCYM with rational $\theta = \frac{n}{s}$.

Symmetry factors $\nu_a!$ are preserved, as is the total number $|\vec{\nu}|$ of sub-modules in any given partition $(\vec{p},\vec{q})$.

$\Rightarrow$ Fluctuation factors in $Z_{p,q}$ invariant under $\theta=0$ Morita

$$\Leftrightarrow \quad a' = \frac{a}{|s|} \quad \text{there.}$$

For any given partition $(\vec{p}, \vec{q})$, let $\nu_a \equiv$ # of submodules in the sequence $0 \leq \dim \mathcal{E}_{p_1, q_1} \leq \dim \mathcal{E}_{p_2, q_2} \leq \dim \mathcal{E}_{p_3, q_3} \leq \dots$ that have the a^{th} least dimension.

$$\Rightarrow \boxed{Z_{p,q} = \sum_{\substack{\text{partitions} \\ (\vec{p}, \vec{q})}} \prod_{a \geq 1} \frac{(-1)^{\nu_a}}{\nu_a!} \left(\frac{g^2 A}{2\pi^2} (p_a - q_a \theta) \right)^{-\nu_a/2} \times \exp\left[-\frac{2\pi^2}{g^2 A} \sum_{k=1}^{|\vec{\nu}|} (p_k - q_k \theta) \left(\frac{q_k}{p_k - q_k \theta} - \frac{q}{p - q\theta} \right)^2 \right]}.$$

~~REMARKS: 1) $\theta = \frac{n}{s}$ $\Rightarrow$ all partitions contain at most $p(+qn$ submodules of $\mathcal{E}_{p,q}$ of dimension $\geq \frac{1}{s}$.~~

~~θ = irrational $\Rightarrow$ no a priori bound on # of submodules in a partition (although it is always finite), and submodules of arbitrarily small dimension can contribute to a partition.~~