Geometrical Methods in Superstring and Superbrane Theories

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NSR formulation

(spinning particles and strings in $D=10$)

worldsheet is a supersurface $\mathcal{M}_W$: $z^M = (\xi, \eta)$

bosonic coordinates — Grassmann-odd $\eta^2 \equiv 0$

embedded into a bosonic target space $\mathcal{M}_{TS}$: $X^{\underline{m}}$, $\underline{m}=0,\dots,9$

$$X^{\underline{m}}(z^M) = x^{\underline{m}}(\xi) + i\eta\,\chi^{\underline{m}}(\xi)$$

GRASSMANN-odd vector describes spin degrees of freedom

- NSR formulation is invariant under worldsheet superdiffeomorphisms $z'^M = z'^M(z)$ which include $\mathcal{M}_W$ bosonic reparametrizations and local worldsheet SUSY $\delta\eta = \varepsilon(\xi)$, $\delta\xi = i\varepsilon(\xi)\eta$

constraints:

$P_{\underline{m}} P^{\underline{m}} = 0$ – masslessness condition (or Virasoro constraints)

$P_{\underline{m}} \chi^{\underline{m}} = 0$ – "Dirac" constraint

- Target-space SUSY appears only upon quantization and the GSO projection

- The NSR formulation is covariantly quantizable

GS - formulation

(superparticles, superstrings & superbranes)

Example: in $N=1$, $D=10$ target superspace

$$x^{\underline{m}}(\xi) \quad (\underline{m}=0,\dots,9); \qquad \theta^{\underline{\alpha}}(\xi) \quad (\underline{\alpha}=1,\dots,16)$$

- reparametrizations $\xi \to \xi'(\xi)$
- target-space SUSY

$$\delta\theta^{\underline{\alpha}} = \epsilon^{\underline{\alpha}}; \qquad \delta x^{\underline{m}} = i\bar{\theta}\Gamma^{\underline{m}}\delta\theta$$

- $\varkappa$-symmetry (de Azcarraga & Lukirski 82; Siegel 83)

$$\delta\theta^{\underline{\alpha}} = P^{\underline{\alpha}}{}_{\underline{\beta}}\,\varkappa^{\underline{\beta}}(\xi), \quad \delta x^{\underline{m}} = -i\bar{\theta}\Gamma^{\underline{m}}\delta\theta$$

$P^{\underline{\alpha}}{}_{\underline{\beta}}$ - is a projector matrix $(\det P = 0)$

For superparticles $\quad P^{\underline{\alpha}}{}_{\underline{\beta}} = \underset{\text{particle momentum}}{p^{\underline{m}}}(\Gamma_{\underline{m}})^{\underline{\alpha}}{}_{\underline{\beta}}$

$$\varkappa_{\alpha} = p_{\alpha\beta}\,\hat{\varkappa}^{\beta} \qquad\qquad p^{\underline{m}}p_{\underline{m}} = 0$$

$\Downarrow$

(infinite) reducibility of $\varkappa$-symmetry

only <u>half</u> (8) of the parameters $\varkappa^{\underline{\alpha}}$ (16) are independent

| this is the source of the covariant quantization problem

- $\varkappa$-symmetry ensures the existence of BPS configurations

Double supersymmetrization

(accumulates properties of both NSR and GS)

The dynamics of superbranes is described by superembedding a worldvolume supersurface $\mathcal{M}_W$ into target superspace $\mathcal{M}_{TS}$

terms of higher orders in η^α

$$X^{\underline{m}}(\xi,\eta^\alpha) = x^{\underline{m}}(\xi) + i\eta^\alpha \chi_\alpha^{\underline{m}}(\xi) + \ldots; \quad \Theta^{\underline{\alpha}}(\xi,\eta) = \theta^{\underline{\alpha}}(\xi) + \eta^\alpha \lambda_\alpha^{\underline{\alpha}}(\xi) + \ldots$$

commuting spinor variables

$\underline{m} = 0, 1, \ldots, 9$
$\underline{\alpha} = 1, \ldots, \underline{16}$ } - target superspace indices

$\alpha = 1, \ldots, \underline{8}$ - the number of Grassmann-odd directions of superworldvolume is half the number of $\Theta^{\underline{\alpha}}$, since our goal is to replace κ-transformations with local worldvolume SUSY

Naive double supersymmetrization produces spinning superparticles and superstrings (Gates et.al. 1986) (Lukierski et al, [illegible]) with double number of physical degrees of freedom in comparison with conventional superstrings.

- κ-symmetry remains an independent symmetry

The superembedding condition

(Tkach, Volkov & D. S., 1988)

The idea is to impose constraints on the superfields $X^{\underline{m}}(\xi,\eta)$ and $\Theta^{\underline{\mu}}(\xi,\eta)$ in such a way that some of their components are expressed in terms of other so that only component fields which describe conventional superbranes remain independent (i.e. $x^{\underline{m}}(\xi)$ and $\theta^{\underline{\mu}}(\xi)$)

From the geometrical point of view this means that one should consider a specific embedding of $\mathcal{M}_W$ into $\mathcal{M}_{TS}$

Target superspace geometry is described by supervielbein one-forms:

$$E^{\underline{A}}(Z) = dZ^{\underline{M}} E_{\underline{M}}{}^{\underline{A}} = (\underbrace{E^{\underline{a}}}_{\text{vector}}, \underbrace{E^{\underline{\alpha}}}_{\text{Grassmann-odd spinor}}), \quad Z^{\underline{M}} = (X^{\underline{m}}, \Theta^{\underline{\mu}})$$

superworldvolume geometry is described by:

$$e^{A}(z) = dz^{M} e_{M}{}^{A}(z) = (\underbrace{e^{a}(\xi,\eta)}_{\text{vector}}, \underbrace{e^{\alpha}(\xi,\eta)}_{\text{Grassmann-odd spinor}}) \quad z^{M} = (\xi, \eta^{\mu})$$

Pullback: $E^{\underline{a}} = e^{a} E_{a}{}^{\underline{a}}(Z(z)) + e^{\alpha}(z) E_{\alpha}{}^{\underline{a}}(Z(z))$

The superembedding condition

$$\boxed{E_{\alpha}{}^{\underline{a}}(Z(\xi,\eta)) = 0}$$

Example: A superparticle in flat $\mathcal{N}=1$, $D=10$ target superspace

The superembedding condition:

$$D_\alpha X^{\underline{m}}(\xi,\eta) - i D_\alpha \bar{\Theta}\Gamma^{\underline{m}}\Theta(\xi,\eta) = 0$$

where $D_\alpha = \frac{\partial}{\partial \eta^\alpha} + i\eta^\alpha \frac{\partial}{\partial \xi}$ - superworldline covariant derivatives

$\{D_\alpha, D_\beta\} = 2i\delta_{\alpha\beta}\partial_\xi$

For components of
$$X^{\underline{m}} = x^{\underline{m}}(\xi) + i\eta^\alpha \chi^{\underline{m}}_\alpha(\xi) + \dots$$
$$\Theta^{\underline{\mu}} = \theta^{\underline{\mu}}(\xi) + \eta^\alpha \lambda_\alpha^{\underline{\mu}}(\xi) + \dots$$

we have:

$$p^{\underline{m}} \equiv \partial_\xi x^{\underline{m}}(\xi) - i\partial_\xi \bar{\theta}\Gamma^{\underline{m}}\theta = \bar{\lambda}_\alpha \Gamma^{\underline{m}} \lambda_\alpha$$

Penrose twistor relation!

summation

superparticle momentum

$$\Downarrow$$

$$\boxed{p^{\underline{m}} p_{\underline{m}} = 0}$$ - the superparticle is massless

$$\chi^{\underline{m}}_\alpha(\xi) = \lambda_\alpha \Gamma^{\underline{m}}\theta$$ - relation between NSR and GS fermionic variables

Hence, only $x^{\underline{m}}(\xi)$ and $\theta^{\underline{\mu}}(\xi)$ are independent dynamical variables.

Local SUSY or irreducible κ-symmetry transformations:

$$\delta\theta^{\underline{\mu}} = \kappa^\alpha(\xi)\lambda^{\underline{\mu}}_\alpha \; ; \quad \delta x^{\underline{m}} = -i\bar{\theta}\Gamma^{\underline{m}}\delta\theta \equiv i\kappa^\alpha \chi^{\underline{m}}_\alpha(\xi)$$

$\alpha = 1,\dots,8$; $\underline{\mu} = 1,\dots,16$

Conclusion

- Superembedding specified by the condition

$$E_\alpha{}^a\left(X(\xi,\eta), \Theta(\xi,\eta)\right) = 0$$

is generic for all known superparticles, superstrings and super-p-branes.

- In some cases the superembedding condition produces only "kinematic" constraints (like $P_m \Gamma^m = 0$) and does not put superbrane dynamics on the mass shell

examples: $N=1$ superparticles and the heterotic string

In these cases worldvolume superfield actions can be constructed

- In other cases, as the supermembrane and the super-5-brane of M-theory ($D=11$ SUGRA) (Bandos, Pasti, Sorokin, Volkov & D.S., 95) the superembedding condition contains all the constraints and the dynamical equations of motion of the superbranes

- The power of superembedding: the full set of equations of motion of the 5-brane was first obtained by this method (Howe & Sezgin, 96)

They were shown to be equivalent to the equations derived from the action principle (Bandos, Lechner, Nurmagambetov, Pasti, Tonin & D.S., 97)
J. Schwarz et. al..

- The superembedding approach explains the geometrical origin of the $\varkappa$-symmetry which turns out to be a specific manifestation of a local worldvolum supersymmetry.

  This allows one to find an irreducible realization of the $\varkappa$-symmetry and provides the geometrical ground for the methods of the covariant quantization of superstrings (Berkovits 1990 - 2002)

Appendix 2 **Massless bosonic relativistic particle**

$$S_{2nd} = \int d\tau \frac{1}{2e(\tau)} \dot{x}^{\underline{m}} \dot{x}^{\underline{n}} \eta_{\underline{m}\underline{n}} ; \quad S_{1st} = \int d\tau \left(P_{\underline{m}} \dot{x}^{\underline{m}} - \frac{e}{2} P_{\underline{m}} P^{\underline{m}} \right)$$

reparametrization invariance $\tau \to f(\tau) \Rightarrow \quad P_{\underline{m}} P^{\underline{m}} = 0$

Spin $\frac{1}{2}$ particle ("NSR" formulation)

$$S = \int d\tau d\eta \left[-\frac{i}{2\mathcal{E}} D X^{\underline{m}} \partial_\tau X^{\underline{n}} \eta_{\underline{m}\underline{n}} \right]$$

$n=1$ local susy on worldline superspace $\quad \delta\eta = \alpha(\tau), \quad \delta\tau = i\alpha(\tau)\eta$

$$\mathcal{E}(\tau,\eta) = e(\tau) + i\eta\Psi(\tau) \qquad D = \frac{\partial}{\partial\eta} + i\eta\frac{\partial}{\partial\tau}$$

$$X^{\underline{m}}(\tau,\eta) = x^{\underline{m}}(\tau) + i\eta\chi^{\underline{m}}(\tau)$$

$$\left\{ \delta x^{\underline{m}}(\tau) = i\alpha(\tau)\chi^{\underline{m}}, \quad \delta\chi^{\underline{m}}(\tau) = \alpha(\tau)\dot{x}^{\underline{m}} \right.$$

constraints

$$P_{\underline{m}} P^{\underline{m}} = 0, \qquad P_{\underline{m}} \chi^{\underline{m}} = 0$$

$\Downarrow$

Dirac equation

Superparticle in N=1 target superspace

$$Z^{M} = (x^{m}, \theta^{\alpha})$$

Space-time N=1 SUSY transformations

$$\delta\theta^{\alpha} = \epsilon^{\alpha}, \qquad \delta x^{m} = i\bar{\theta}\Gamma^{m}\delta\theta$$

$$S = \int d\tau \, \frac{1}{2e} E_{\tau}^{m} E_{\tau}^{n} \eta_{mn}$$

or

$$S = \int d\tau \left[P_{m} E_{\tau}^{m} - \frac{e(\tau)}{2} P_{m} P^{m} \right] \qquad E_{\tau}^{m} \equiv$$

$$E^{m} = dx^{m} - i\, d\bar{\theta}\Gamma^{m}\theta = d\tau\left(\partial_{\tau} x^{m} - i\,\partial_{\tau}\bar{\theta}\Gamma^{m}\theta\right)$$

(superinvariant one-form)

Local fermionic $\mathcal{K}$-symmetry

$$\delta_{\kappa}\theta^{\alpha} = E_{\tau}^{m}\Gamma_{m}{}^{\alpha}{}_{\beta}\,\kappa^{\beta}(\tau), \qquad \delta_{\kappa} x^{m} = -i\bar{\theta}\Gamma^{m}\delta_{\kappa}\theta$$

$$\delta_{\kappa} e(\tau) = 4i\,\dot{\bar{\theta}}\,\kappa(\tau)$$

{ κ-symmetry is infinite reducible

In $D = 3, 4, 6, 10$ κ^{α} has $2(D-2)$ components

but only half of them, i.e. $(D-2)$, contribute to the variations of the fields

Introduction to the geometrical description of superbrane dynamics

hep-th/9906142, Phys. Reports 329 (2000) 1

0105102

Geometrical approach in which superbrane dynamics is formulated in terms of the embedding a worldvolume <u>super</u>surface into target <u>superspace</u>.

Main features

- unifies the Neveu-Schwarz-Ramond and the Green-Schwarz formulation of the superstring
- explains the origin of fermionic κ-symmetry of the GS-formulation as standard local supersymmetry of the worldvolume of the superbrane
  (D.S., Tkach and Volkov, 1988)
- thus, solves the problem of infinite reducibility of κ-symmetry by providing one with its irreducible SUSY realization
- progress in covariant quantization of GS-superstring
  (Berkovits
  1990 – [illegible])
- a universal and powerful method for deriving equations of motion for new superbranes, such as the M-theory 5-brane (Howe & Sezgin, 1996)