Nonabelian Gauge Theories on Noncommutative Spaces

1. NCGT via S-W map

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Noncommutative spaces and
$\star$-products

Coordinates $\hat{x}^i$ + relations

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$$

or more generally

$$[\hat{x}^i, \hat{x}^j] = i f^{ij}_k \hat{x}^k \qquad \text{Lie algebra}$$

$$[\hat{x}^i, \hat{x}^j] = i C^{ij}_{kl} \hat{x}^k \hat{x}^l$$

$$\vdots$$

Noncommutative $\mathbb{R}^4$ (Moyal - Weyl)

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$$

as **deformation quantization** of the Poisson bracket

$$\{x^i, x^j\} = \theta^{ij}$$

$$(f \star g)(x) = e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x) g(y)\Big|_{x=y}$$

$$x^i \star x^j = \frac{i}{2}\theta^{ij}$$

$$[x^i \overset{\star}{,} x^j] = i\theta^{ij} \qquad \mathcal{A}_\star$$

Lie algebra case $\quad x^i \sim t^i$

$$(f * g)(t) = e^{\frac{i}{2} t^i g_i\left(i\frac{\partial}{\partial t'}, i\frac{\partial}{\partial t''}\right)} f(t') g(t'')\Big|_{\substack{t'=t \\ t''=t}}$$

$$e^{ik_i \hat{t}^i} e^{ip_j \hat{t}^j} = e^{i\{(k_i+p_i)+\frac{1}{2}g_i(k,p)\}\hat{t}^i}$$

$\mathcal{A}_t \sim$ universal enveloping algebra

$\mathcal{A}_x \otimes \mathcal{A}_t$

Enveloping algebra valued gauge fields.

matter fields $\quad \hat{\psi}(x) \in \mathcal{A}_x \otimes V$ - representation space of Lie algebra g

NC gauge transformation

$$\delta_{\hat{\Lambda}} \hat{\psi}(x) = i \underset{\substack{\cap \\ \mathcal{A}_x \otimes \mathcal{A}_t}}{\hat{\Lambda}(x)} * \hat{\psi}(x)$$

$* = *_x \otimes$ action of $\mathcal{A}_t$ in V (matrix multiplication)

$$\delta_{\hat\Lambda} \hat{A}_\mu = \partial_\mu \hat\Lambda + i[\hat\Lambda \overset{*}{,} \hat{A}_\mu]$$

$A_\mu \in \mathcal{A}_x \otimes \mathcal{A}_t$

$[\overset{*}{,}]$ ~ commutator in $\mathcal{A}_x \otimes \mathcal{A}_t$

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu \overset{*}{,} \hat{A}_\nu]$$

$$\delta_\Lambda \hat{F}_{\mu\nu} = i[\hat\Lambda \overset{*}{,} \hat{F}_{\mu\nu}]$$

$$D_\mu \hat\Psi = \partial_\mu \hat\Psi - i \hat{A}_\mu * \hat\Psi$$

$*_x \otimes$ action in V

The same degrees of freedom as ordinary gauge theory via Seiberg-Witten map.

S–W

$\hat{A}(A, \theta)$

$\hat{\Lambda}(\Lambda, A, \theta)$

$\hat{\Psi}(\Psi, A, \theta)$

any Lie algebra
any representation

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda + i[\Lambda, A_\mu]$$

$$\delta_\Lambda \Psi = i \Lambda \Psi$$

$$\Rightarrow \quad \delta_{\hat\Lambda} \hat{A}_\mu = \partial_\mu \hat\Lambda + i[\hat\Lambda \overset{*}{,} \hat{A}_\mu]$$

$$\delta_{\hat\Lambda} \hat\Psi = i \hat\Lambda * \hat\Psi$$

$$\hat{A}_\xi[A, \theta] = A_\xi + \frac{1}{4}\theta^{\mu\nu}\{A_\nu, \partial_\mu A_\xi\} + \frac{1}{4}\theta^{\mu\nu}\{F_{\mu\xi}, A_\nu\} + O(\theta^2)$$

$$\hat\Lambda[\Lambda, \theta, A] = \Lambda + \frac{1}{4}\theta^{\mu\nu}\{\partial_\mu \Lambda, A_\nu\} + O(\theta^2)$$

$$\hat\Psi[\Psi, \theta, A] = \Psi + \frac{1}{2}\theta^{\mu\nu} A_\nu \partial_\nu \Psi + \frac{i}{8}\theta^{\mu\nu}[A_\mu, A_\nu]\Psi + O(\theta^2)$$

not unique

NC YM action

$$\hat{S} = \int d^4x \, \frac{-1}{2g^2} \mathrm{Tr}\, \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} + \bar{\hat{\Psi}} * i \hat{\not{D}} \hat{\Psi}$$

$\hat{F}_{\mu\nu} \in$ enveloping algebra

$$\hat{F}_{\mu\nu} = \sum_{s=1}^{\infty} \sum_{a_1, \ldots, a_s} \mathcal{F}_{\mu\nu}^{(a_1, \ldots, a_s)}(\theta, \partial, A) T^{a_1} \ldots T^{a_s}$$

Tr ???

In general

$$\frac{1}{g^2} \mathrm{Tr}\, \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} = \sum_{\rho} c_\rho \mathrm{Tr}\, \rho(\hat{F}_{\mu\nu}) * \rho(\hat{F}^{\mu\nu})$$

+ condition (from $\theta \to 0$)

$$\frac{1}{g^2} = \sum_{\rho} c_\rho \mathrm{Tr}(\rho(T^a)\rho(T^a))$$

Example   Tr — trace in the fundamental repr.

$$-\frac{1}{4}\mathrm{Tr}\int \hat{F}_{ij} * \hat{F}^{ij}\, dx =$$

$$= -\frac{1}{4}\,\mathrm{Tr}\int F_{ij}\cdot F^{ij}\,dx + \frac{1}{8}\,\theta^{kl}\,\mathrm{Tr}\int F_{kl}\,F_{ij}\cdot F^{ij}\,dx$$

$$-\frac{1}{2}\,\theta^{kl}\int F_{ik}\,F_{jl}\,F^{ij}\,dx + \cdots$$

$$\int \bar{\hat{\psi}} * (\gamma^{i}\hat{D}_i - m) * \hat{\psi}\, dx$$

$$= \int \psi\,(\gamma^{i}D_i - m)\,\psi\,dx$$

$$-\frac{1}{4}\,\theta^{kl}\int \psi\, F_{kl}\,(\gamma^{i}D_i - m)\,\psi\,dx$$

$$-\frac{1}{2}\,\theta^{kl}\int \psi\,\gamma^{i}F_{ik}\,D_l\,\psi\,dx + \cdots$$

$$\int dx\ f*g*h = \int dx\,(f*g)\cdot h$$

$$\int f*g = \int f\cdot g$$

(7)

Consistency (cocycle) condition

$$[\hat{\Lambda}_\alpha[A] \stackrel{*}{,} \hat{\Lambda}_\beta[A]] + i\delta_\alpha \hat{\Lambda}_\beta[A] - i\delta_\beta \hat{\Lambda}_\alpha[A] = \hat{\Lambda}_{[\alpha,\beta]}[A]$$

$$\hat{\Lambda}_\alpha[A] \equiv \hat{\Lambda}(\alpha, \theta, A)$$

$$\hat{\Lambda}_\Lambda[A] = \Lambda + \frac{1}{2}\theta^{ij}\{A_j, \partial_i\Lambda\}_c + O(\theta^2)$$

$$\{P, Q\}_c = c \cdot P \cdot Q + (1-c) Q \cdot P$$

c - arbitrary function on space time

- most general solution (up to field redefinition of A, Λ)

hermiticity $c = \frac{1}{2}$ + purely imaginary function

solving for S-W map: possible strategy take $\hat{\Lambda}_\Lambda[A]$ and use transformation properties to find $\hat{A}$ and $\hat{\psi}$

⑧

Standard Model

$$G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$$

$$V_\nu = g' A_\nu(x) Y + g \sum_{a=1}^{3} B_{\nu a}(x) T_L^a + g_s \sum_{b=1}^{8} G_{\nu b}(x) T_S^b$$

$$\Lambda = g' \alpha(x) Y + g \sum_{a=1}^{3} \alpha_a^L(x) T_L^a + g_S \sum_{s=1}^{8} \alpha_b^S(x) T_S^b$$

$$\psi_L^{(i)} = \begin{pmatrix} L_L^{(i)} \\ Q_L^{(i)} \end{pmatrix} \qquad \psi_R^{(i)} = \begin{pmatrix} e_R^{(i)} \\ u_R^{(i)} \\ d_R^{(i)} \end{pmatrix}$$

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

$$V_\nu, \psi^{(n)} \xrightarrow{S-\alpha} \hat{V}, \hat{\psi}^{(n)}$$

the same particle spectrum

8a)

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$U(1)_Q$
e_R	1	1	-1	-1
$L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	1	2	$-1/2$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
u_R	3	1	$2/3$	$2/3$
d_R	3	1	$-1/3$	$-1/3$
$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	3	2	$1/6$	$\begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$
$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	1	2	$1/2$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
B^i	1	3	0	$(\pm 1, 0)$
A	1	1	0	0
G^a	8	1	0	0

$$Q = T_3 + Y$$

$B^i \quad i = +, -, 3$

Non-commutative Higgs $\hat{\phi}$
more complicated.

ϕ commutes with classical $U(1)$ and
$SU(3)$ gauge parameters
but $\hat{\phi}$ doesn't commute with
noncommutative $U(1)$ and $SU(3)$ gauge
parameters. For gauge invariant
Yukawa couplings we need

$$\hat{\phi} = \hat{\phi}[\phi, A, -A'] =$$

$$= \phi + \frac{1}{2}\theta^{\mu\nu} A_\mu \left(\partial_\mu \phi - \frac{i}{2}(A_\mu \phi - \phi A'_\mu\right)$$

$$+ \frac{1}{2}\theta^{\mu\nu}\left(\partial_\mu \phi - \frac{i}{2}(A_\mu \phi - \phi A'_\mu\right) A'_\nu + O(\theta^2)$$

$$\delta_{\Lambda,\Lambda'}\hat{\phi} = i\hat{\Lambda} * \hat{\phi} - i\hat{\phi} * \hat{\Lambda}'$$

$$\hat{D}_\mu \hat{\phi} = \partial_\mu \hat{\phi} - i\hat{A}_\mu * \hat{\phi} + i\hat{\phi} * \hat{A}'_\mu$$

$$S = \int d^4x \sum_{i=1}^{3} \hat{\bar{\Psi}}_L^{(i)} * i \hat{\not{D}} \hat{\Psi}_L^{(i)}$$

$$+ \int d^4x \sum_{i=1}^{3} \hat{\bar{\Psi}}_R^{(i)} * i \hat{\not{D}} \hat{\Psi}_R^{(i)}$$

$$- \int d^4x \frac{1}{2g'} \mathrm{tr}_1 \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} - \int d^4x \frac{1}{2g} \mathrm{tr}_2 \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu}$$

$$- \int d^4x \frac{1}{2g_s} \mathrm{tr}_3 \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} + \int d^4x\, \rho_0(\hat{D}_\mu \hat{\phi})^\dagger * \rho_0(\hat{D}^\mu \hat{\phi})$$

$$+ \int d^4x \Big(- \sum_{i,j=1}^{3} W^{ij} \Big((\hat{\bar{L}}_L^{(i)} * \rho_L(\hat{\phi}) * \hat{e}_R^{(j)} + \hat{\bar{e}}_R^{(i)} * \rho_L(\hat{\phi})^\dagger * \hat{L}_L^{(j)} \Big)$$

$$- \sum_{i,j}^{3} G_u^{ij} \Big((\hat{\bar{Q}}_L^{(i)} * \rho_{\bar{Q}}(\hat{\bar{\phi}}) * u_R^{(j)} + \hat{\bar{u}}_R^{(i)} * (\rho_{\bar{Q}} \hat{\bar{\phi}})^\dagger * Q_L^{(j)} \Big)$$

$$- \sum_{i,j}^{3} G_d^{ij} \Big(\hat{\bar{Q}}_L^{(i)} * \rho_Q(\hat{\phi}) * \hat{d}_R^{(j)} + \hat{\bar{d}}_R^{(i)} * (\rho_Q \hat{\phi})^\dagger * Q_L^{(j)} \Big) \Big)$$

$$\bar{\phi} = i\tau_2 \phi^*$$

Minimal modification from the ~~commutative~~ case $U(1)$ trace in the representation

$$Y = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Tr - sum over $U(1)$, $SU(2)$ and $SU(3)$ sectors

tr_2 and tr_3 - the usual ones.

$$\rho_L(\hat{\Phi}(\phi, V_\mu, V_\nu')) = \hat{\Phi}[\phi, -\tfrac{1}{2}g' A_\mu + g B_\mu^a T_L^a, g' A_\nu]$$

$$\rho_Q \hat{\Phi} = \hat{\Phi}[\phi, \tfrac{1}{6}g' A_\mu + g B_\mu^a T_L^a + g_s G_\mu^a T_S^a, \tfrac{1}{3}g' A_\nu - g_s G_\nu^a T_S^a]$$

$$\rho_{\bar{Q}} \hat{\Phi} = \hat{\Phi}[\phi, \tfrac{1}{6}g' A_\mu + g B_\mu^a T_L^a + g_s G_\mu^a T_S^a, -\tfrac{2}{3}g' A_\nu - g_s G_\nu^a T_S^a]$$

$$\rho_0 \hat{\phi} = \hat{\phi}[\phi, \tfrac{1}{2}g' A_\mu + g B_\mu^a T_L^a, 0]$$

use S-W for $\hat{V}_\mu$, $\hat{\psi}^{(n)}$, $\hat{\Phi}$

and plug in to S to obtain

S_{NCSM} up to the 1-st order in θ.

Minimal version of NCSM.

other possible choice

- weighted sum over fermion representations

this gives e.g. triple photon vertex absent in the minimal version.

GUT inspired NCSM

Some (advantages) properties

- $SU(3) \times SU(2) \times U(1)$, zeroth order in θ reproduces SM
- the ordinary gauge & matter fields
- Higgs mechanism and Yukawa sector are fine
- QED not changed, photon massless
- gauge bosons of different gauge groups decouple
- no self-interacting vertices of $U(1)_Y$ gauge boson (possible in nonminimal versions) but aditional vertices with $SU(3)_C$ and $SU(2)_L$ gauge bosons
- new vertices connecting gauge bosons of different gauge groups
  e.g. $SU(3)_C$ and $U(1)_Y$ to quarks – parity violation in NCQED
- no couplings of Higgs boson ~~to the~~ photon but new couplings ~~between~~ the Higgs and electroweak ~~gauge~~ bosons
- richer flavour physics

expansion in θ - expansion in the transfered momentum - low energy effective actions

Some remaining problems

- renormalisation
(freedom in S-W map important)
infinitely many counter terms
- UV/IR has to be reconsidered
- temporal NC - unresolved problems with unitarity

Possible phenomenology (including non-minimal versions)

- SM forbidden three level decays

  $Z \to 2\gamma$

  $\bar{Q}Q \to 2\gamma$

- SM spin forbidden decays from flavor changing neutral currents

  $K \to \pi\gamma$

  $B \to K\gamma$

- θ - contributions to scatterings and annihilations

  Moller, Bhabha, $\gamma\gamma \to \gamma\gamma$

  $\ell^+\ell^- \to \gamma\gamma$

Lower bound for NC scale from "high energy" physics phenomenology based on constant θ

$$\Lambda_{NC} \sim 1-2 \text{ TeV}$$

LHC $\sim 10^{11}-10^{12}$ $\bar{Q}Q$/year should be enough to observe few events

three level $L_{Z\gamma\gamma}$ Behr, Deshpande, Duplančić, Schupp, Trampetić, Wess

$$\theta^{kl}\Big[2(-\partial_i Z_k + \partial_k Z_i)\partial_j A_l(\partial^i A^j - \partial^j A^i)$$

$$+ (\partial_i A_k \partial_j A_l + \partial_k A_i \partial_l A_j - 2\partial_k A_i A_j A_l)(\partial^i Z^j - \partial^j Z^i)$$

$$+ (-2\partial_k Z_i \partial_l A_j + 2\partial_j Z_l \partial_k A_i + 2\partial_i Z_j \partial_k A_l + \partial_k Z_l \partial_i A_j)$$

$$\times (\partial^i A^j - \partial^j A^i)$$

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Covariant functions, coordinates

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A_x associative unital algebra generated by x^i, modulo relations

Ex.

$$x^i x^j - x^j x^i = \theta^{ij}$$
$$x^i x^j - x^j x^i = C^{ij}_k x^k$$
$$\vdots$$

M - left module (finitely generated projective)

ψ - M - matter fields

Gauge transformation

$$\psi \longmapsto \Lambda . \psi \qquad \Lambda \in A_x \text{ gauge parameter}$$

However $f \in A_x$

$$f \cdot \psi \longmapsto f \cdot \Lambda \cdot \psi \neq \Lambda \cdot f \cdot \psi$$

Covariant functions

$$Df = f + f_A \quad \text{such that}$$

$$Df \longmapsto \Lambda \cdot Df \cdot \Lambda^{-1}$$

$$\Rightarrow \quad f_A \mapsto \Lambda\cdot[f,\Lambda^{-1}] + \Lambda\cdot f_A\cdot\Lambda^{-1}$$

Notation $\quad f_A = A(f)$

$$F(f,g) = [Df, Dg] - D[f,g]$$

Ex. $\quad [x^i, x^j] = i\theta^{ij} \qquad \theta^{ij}$ - const. invertible

$$f * g = f\, e^{\frac{i}{2}\theta^{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j}}\, g \qquad \text{Moyal-Weyl}$$

$$Dx^i = x^i + x^i_A = x^i + \theta^{ij}\hat{A}_j$$

$$\hat{A}_j \mapsto i\Lambda * \partial_j(\Lambda^{-1}) + \Lambda * \hat{A}_j * \Lambda^{-1}$$

$$\theta^{-1}_{ik}\theta^{-1}_{jl}F(x^k, x^l) = \hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i - i\,\boxed{[\hat{A}_i \overset{*}{,} \hat{A}_j]}$$

String theory motivation (SW)

Open string σ-model with background term

$$S_B = \frac{1}{2i} \int_{Disc} B_{ij} \varepsilon^{ab} \partial_a X^i \partial_b X^j$$

B- constant, nondegenerate
correlation functions on ∂D
in the decoupling limit $(g \to 0, \alpha' \to 0)$

$$\langle f_1(x(\tau_1)) \dots f_n(x(\tau_n)) \rangle_B =$$

$$= \int dx \; f_1 * f_2 * \dots * f_n \qquad (\tau_1 < \dots < \tau_n)$$

$*$ Moyal-Weyl $\quad \theta^{ij} = B_{ij}^{-1}$

Perturbation $\quad B \to B + da$

$$S_B \to S_B - i \int_{\partial D} d\tau \; a_i(x(\tau)) \partial_\tau x^i(\tau)$$

Naive gauge invariance

$$\delta a_i = \partial_i \lambda$$

Regularization

e.g. Pauli-Villars respects g.s.

Point-splitting doesn't

but still "noncommutative" g.s.

$a_i \to \hat{A}_i$

$$\hat{\delta}_{\hat{\lambda}} = \partial_i \hat{\lambda} + i \hat{\lambda} * \hat{A}_i - i \hat{A}_i * \hat{\lambda}$$

If the theory is independent on the regularization scheme a map relating ordinary g.s. and the noncommutative g.s. is to be expected SW map

$\hat{A}(a, \theta)$

$\hat{\lambda}(\lambda, a, \theta)$ (linear in λ)

such that

$$\hat{A}(a + \delta_\lambda a) = \hat{A}(a) + \hat{\delta}_{\hat{\lambda}} \hat{A}(a) + O(\lambda^2)$$

On the other hand

$$-i \int_{\partial D} d\tau \, a_i(x) \partial_\tau x^i(\tau) =$$

$$= \frac{i}{2} \int_D F_{ij} \, \varepsilon^{ab} \partial_a x^i \partial_b x^j \qquad F = da$$

$$B \to B + da$$

$*_{\text{Moyal Weyl}} \longrightarrow *'$

$*'$ is deformation quantization

of $\theta' = \theta - \theta F \theta + \theta F \theta F \theta - \cdots$

$(B+F)^{-1} = \theta'$

SW should somehow relate

$*$ and $*'$ (commutative versus noncommutative description of D-branes)

Moser lemma (formal)

$\theta^{ij}(x)$ - Poisson structure any

Jacobi $\theta^{ik} \partial_k \theta^{jl}$ + cyclic permutations $= 0$

$$\theta_t = \sum (-t)^n \theta (F\theta)^n \qquad \theta_{t=0} = \theta(x)$$

$$\{f,g\}_t = \theta_t^{ij} \partial_i f \partial_j g$$

$$\{\,,\,\}_{t=0} = \{\,,\,\} \qquad \{\,,\,\}_{t=1} = \{\,,\,\}'$$

$$\chi_t = \theta_t^{ij} a_j \partial_i$$

ρ_t - flow generated by $\chi(t)$

$$\rho_a^* = e^{\partial_t + \chi_t} e^{-\partial_t} \Big|_{t=0}$$

$$\rho_a^* \{f, g\}' = \{\rho_a^* f, \rho_a^* g\}$$

What happens if $a \to a + \delta_\lambda a = a + d\lambda$

$$\rho_{a+d\lambda}^*(f) = \rho_a^*(f) + \{\rho_a^*(f), \tilde{\lambda}\} + \dots$$

$$\tilde{\lambda}(\lambda, a) = \sum_{n=0}^{\infty} \frac{(\chi(t) + \partial_t)^n}{n!} \lambda \Big|_{t=0}$$

$\rho_a^*(f)$ - covariant function

$$\rho_a^*(f) = f + A(f)$$

A - differential operator $A(a, \theta)$

$a \to A(a, \theta)$ is a <u>semiclassical version of SW map</u>

Kontsevich's formality map gives the quantum version of the Moser lemma

$$\hat{\Psi}_t = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{n!} U_{n+1}(\Psi_t, \theta_t, \dots \theta_t) \quad \text{- diff operator}$$

$$\hat{\lambda} = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{n!} U_{n+1}(\lambda, \theta, \dots, \theta) \quad \text{- function}$$

$$\mathcal{D}_a = e^{\hat{\Psi}_t + \partial_t} e^{-\partial_t} \Big|_{t=0}$$

$$\hat{\Lambda}(\lambda, a) = \sum_{n=0}^{\infty} \frac{(\hat{\Psi}_t + \partial_t)^n}{(n+1)!} \hat{\lambda} \Big|_{t=0}$$

$$\mathcal{D}_a(f *' g) = \mathcal{D}_a f * \mathcal{D}_a g$$

$$\mathcal{D}_{a+d\lambda}(f) = \mathcal{D}_a f + \frac{i}{\hbar}[\hat{\Lambda} \overset{*}{,} \mathcal{D}_a f]$$

$$\mathcal{D}_a(f) = f + \hat{A}(f)$$

$\hat{A}(a, \theta)$ - SW map

$$\begin{tikzcd}
\theta \arrow[r, "\text{Kontsevich}"] \arrow[d, "\text{Moser}"'] & * \arrow[d, "\text{Seiberg-Witten}"] \\
\theta' \arrow[r, "\text{Kontsevich}"'] & *'
\end{tikzcd}$$

SW map (locally $F = da$)
is an equivalence of $*$ and $*'$
globally it is Morita equivalence

$*$ and $*'$ describe topological
open strings in the same closed
string background (B-field) but
in a different open string bck. (F-field)

$$* \overset{\text{Morita}}{\sim} *'$$
$$\Downarrow$$
the same K-theory

Important K-theory classifies D-branes

Finite gauge transformation

$$g \xrightarrow{SW} G_g[a] \qquad g = g_1 \cdot g_2$$

1- cocycle condition $\quad a_g = a + i g d g^{-1}$

$$G_{g_1}[a_{g_2}] * G_{g_2}[a] = G_{g_1 \cdot g_2}[a]$$

$$\mathcal{D}_{a_g}(f) = G_g[a] * \mathcal{D}_a(f) * G_g^{-1}[a]$$

Consider a good covering U_i of a Poisson manifold

a_i - connection on a line bundle g_{ij}

$$\mathcal{D}_i \equiv \mathcal{D}_{a_i} \quad , \quad G_{ij} \equiv G_{g_{ij}}[a_j]$$

$$G_{ij} * G_{jk} * G_{ki} = 1 \quad \text{on } U_i \cap U_j \cap U_k$$

$$G_{ij} * G_{ji} = 1 \quad \text{on } U_i \cap U_j$$

$\mathcal{D}_i$ on U_i an equivalence of $*$ and $*'$

$*$ and $*'$ are Morita equivalent.

Sections $\Psi = (\Psi_k)$, $\Psi_k \in C_c^\infty(M)[[\hbar]]$

$$\Psi_j = G_{jk} * \Psi_k \quad \text{on } U_j \cap U_k$$

$\mathcal{E}$ - space of all sections

$\mathcal{E}_A$ - right module structure of $A = (C^\infty(M)[[\hbar]], *)$

$$\Psi . f = (\Psi_k * f)$$

${}_{A'}\mathcal{E}$ - left module structure of $A' = (C^\infty(M)[[\hbar]], *')$

$$f . \Psi = (D_k(f) * \Psi_k)$$

$$f . (g . \Psi) = (f *' g) . \Psi$$

${}_{A'}\mathcal{E}_A$ (A', A) - bimodule.

Connection (noninvariant) Hochschild complex (C^p, d).

p-cochains $\leftrightarrow$ p-forms

$C^p \cup C^q$ - cup product

$\nabla : \mathcal{E} \otimes_A C^p \to \mathcal{E} \otimes_A C^{p+1}$

Connection ∇ uniquely defined by the first component

$$\nabla : \mathcal{E} \otimes_A C^0 \equiv \mathcal{E} \to \mathcal{E} \otimes C^1$$

Extension by graded Leibnitz rule

$\sum_i h_i *' h_i = 1$ h_i - formal power series

$P_{ij} = \mathcal{D}_i(h_i) * G_{ij} * \mathcal{D}_j(h_j)$

$P_{ij} = \sum P_{ik} * P_{kj}$ Serre-Swan

$\mathcal{E}_A$ right finitely generated projective A-module $\mathcal{E}_A = PA^n$

Similarly ${}_{A'}\mathcal{E} = A^n P'$ t.g. left projective ~~A'-module~~

Morita equivalence

A, A' Morita equivalent iff they have the same representation categories.

iff There exist two equivalence bimodules ${}_{A'}\mathcal{E}_A$ and ${}_A\bar{\mathcal{E}}_{A'}$ such that

$${}_{A'}\mathcal{E} \otimes_A \bar{\mathcal{E}}_{A'} \cong {}_{A'}A'_{A'} \text{ and } {}_A\bar{\mathcal{E}}_{A'} \otimes \mathcal{E}_A \cong {}_A A_A$$

Another possible def. of sections

$$\bar{\Psi} = \bar{\Psi}_k, \quad \bar{\Psi}_k = \bar{\Psi}_j * G_{jk}$$

this gives ${}_A\bar{\mathcal{E}}_{A'}$

${}_A\bar{\mathcal{E}} \cong A^n P$ P - the same as before.

$\mathrm{Pic}(C^\infty(M)) = \{$ equiv. classes of $(C^\infty(M), C^\infty(M))$ bimodules $\mathcal{E}\}$

$\mathrm{Pic}(C^\infty(M)) = \mathrm{Pic}(M)$ – equiv. classes of line bundles

$\mathrm{Pic}(C^\infty(M)) = H^2(M, \mathbb{Z})$

Kontsevich

equiv. classes $[*]$ $\longleftrightarrow$ equiv. classes $[\theta]$

Action of $\mathrm{Pic}(C^\infty(M))$ on $*$

$[*]$ corresponding to $[\theta]$

$\downarrow$

$[*']$ corr. to $[\theta']$ $\quad \theta' = \theta(1+F\theta)^{-1}$

(modulo global isomorphisms)

$[*] \overset{\text{Morita}}{\Longleftrightarrow} [*']$ iff they are on the same orbit of $\mathrm{Pic}(C^\infty(M))$

SCHUPP, WESS, JURCO (1)

NC line bundles

(M, θ) Poisson manifold

$$[\theta, \theta] = 0$$

$$\theta = \frac{1}{2}\theta^{ij}\partial_i \wedge \partial_j \qquad \theta^{ij}\partial_j \theta^{kl} + \dots = 0$$

$$F = \frac{1}{2} f_{ij}\, dx^i \wedge dx^j \qquad dF = 0$$

$$\theta_t \qquad t \in [0,1]$$

$$\theta_t : T^*M \to TM$$
$$F : TM \to T^*M$$

$$\partial_t \theta_t = -\hbar\, \theta_t F \theta_t$$

$$\theta_t = \theta (1 + t\hbar F\theta)^{-1}.$$

$$[\theta_t, \theta_t] = 0 \qquad \theta_0 = \theta, \ \theta_1 = \theta'$$

Kontsevich

$$f *_t g = \sum \frac{(i\hbar)^n}{n!} U_n(\theta_t, \dots, \theta_t)(f, g)$$

$$*_0 = * \qquad *_1 = *'$$

locally $\quad F = da$

$$D_{[a]} = e^{a *_t + \partial_t} e^{-\partial_t} \Big|_{t=0}$$

$$a_{*t} = \sum_{n=0}^{\infty} \frac{(i\hbar)^{n+1}}{(n+1)!} U_{n+1}(a_{\theta_t}, \theta_t, \dots, \theta_t)$$

$$a_{\theta_t}(f) = \theta_t(a, df) \qquad a_{\theta_t} = \theta_t^{ij} a_i \partial_j$$

$$\delta_\lambda a = d\lambda$$

$$\delta_\lambda D_{[a]}(f) = i\left[\Lambda_\lambda[a] \overset{*}{,} D_{[a]}(f)\right]$$

$$\Lambda_\lambda[a] = \sum_{n=0}^{\infty} \frac{(a_{*t} + \partial_t)^n}{(n+1)!} \tilde{\lambda}\Big|_{t=0}$$

$$\tilde{\lambda} = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{(n+1)!} U_{n+1}(\lambda, \theta_t, \dots, \theta_t)$$

$$D_{[a]}(f *' g) = D_{[a]}(f) * D_{[a]}(g)$$

$*'$ def. of $\theta' = \theta(1 + \hbar F\theta)^{-1}$

$\omega = \omega + \hbar F$ if $\theta^{-1} = \omega$

$*'$ - globally defined

Finite gauge transformations

$a_g = a + i g dg^{-1}$

$$D_{[a_g]} \circ D^{-1}_{[g]}(f) = G_g[a] * f * G_g^{-1}[a]$$

1-cocycle

$$G_{g_1}[a_{g_2}] * G_{g_2}[a] = G_{g_1 \cdot g_2}[a]$$

$\{U_i\}$ good covering

commutative line bundle

$$g_{ij} g_{jk} = g_{ki} \qquad \text{on } U_i \cap U_j$$

$$g_{ii} = 1$$

Sections $\quad \psi_j = g_{jk} \psi_k$

$$G^{ij} \equiv G_{g_{ij}}[a_j]$$

Cocycle condition gives

$$\left.\begin{aligned} G^{ij} * G^{jk} &= G^{ik} \\ G^{ij} * G^{ji} &= 1 \end{aligned}\right\} \text{Definition}$$

$+ \quad \mathcal{D}^i = \mathcal{D}_{[a_i]}$

$$G^{ij} * \mathcal{D}^j(f) = \mathcal{D}^i(f) * G^{ij}$$

$$f *' g = \mathcal{D}^{i-1}(\mathcal{D}^i(f) * \mathcal{D}^i(g))$$

$*'$ defined globally

$*'$ – locally equivalent to $*$

$*'$ – globally Morita equivalent to $*$

Sections $\quad \psi^j = G^{jk} * \psi^k \qquad \mathcal{E}$

$$L = \{G^{ij}, \mathcal{D}^i, *\}$$

$L_1 \sim L_2 \qquad G_1^{ij} = H^i * G_2^{ij} * H^{j-1}$

$$D_1^i = Ad_* H^i \circ D_2^i$$

$$L_1 \otimes L_2 = L_{21} = \{D_1^i(G_2^{ij}) * G_1^{ij},\ D_1^i \circ D_2^i,\ *\}$$

$\otimes$ sections $\qquad D^i(\psi_2^i) *_1 \psi_1^i$

${}_{A'}\mathcal{E}_A \qquad A = (C^\infty(M), *)$

$$A' = (C^\infty(M), *')$$

$$\psi^i \to \psi^i * f$$

$$\psi^i \to \mathcal{D}^i(f) * \psi^i$$

– finitely generated right A-module

left A'-module

Hochschild complex for A

$C^p = \mathrm{Hom}_{\mathbb{C}}(A^{\otimes p}, A)$ d - Hochschild

$\nabla : \mathcal{E} \otimes_A C^p \to \mathcal{E} \otimes C^{p+1}$ contravariant connection

$$(\nabla\psi)(f) = \left(D^i(f) * \psi^i - \psi^i * f \right)$$

$$(\nabla^2\psi)(f,g) = \left(D^i(f *' g - f * g) * \psi^i \right)$$

tensor $\nabla_{12}\psi_{12}^i = D_1^i(\nabla_2\psi_2^i) *_1 \psi_1^i$

$$+ D_1^i(\psi_2^i) *_1 \nabla_1 \psi_1^i$$

$$\nabla_{12} = \nabla_1 + D_1(\nabla_2)$$

$* \overset{\text{Morita}}{\sim} *'$

$\Rightarrow$ up to a global isomorphism related by the action of

$P(M) \simeq H^2(M, \mathbb{Z})$

$F \in H^2(M, \mathbb{Z})$

$\theta' = \theta(1 + \hbar F\theta)^{-1}$

if $F = da$,

$$G^{ij} = H^{i-1} * H^j$$

$$D = Ad_* H^i \circ D^i \quad \text{- global equivalence}$$

Gerbes

U_α - good covering

2-cocycle in Čech

$$\lambda_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\beta \to U(1)$$

$$\lambda_{\alpha\beta\gamma} = \lambda^{-1}_{\beta\alpha\gamma} = \lambda^{-1}_{\alpha\gamma\beta} = \lambda^{-1}_{\gamma\beta\alpha}$$

$$\delta\lambda = \lambda_{\beta\gamma\delta}\,\lambda^{-1}_{\alpha\gamma\delta}\,\lambda_{\alpha\beta\delta}\,\lambda^{-1}_{\alpha\beta\gamma} = 1$$

on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

trivial gerbe (coboundary)

$$\lambda_{\alpha\beta\gamma} = h_{\alpha\beta}\, h_{\beta\gamma}\, h_{\gamma\alpha}$$

$h_{\alpha\beta}(h'_{\alpha\beta})^{-1}$ - line bundle

Each gerbe is locally trivial

Fix U_0 $\quad h_{\beta\gamma} = \lambda_{0\beta\gamma}$

U_α – any covering

line bundle $l_{\alpha\beta}$ on each $U_\beta \otimes U_\alpha$

1. $l_{\alpha\beta} \cong l_{\beta\alpha}$

2. $l_{\alpha\beta} \otimes l_{\beta\gamma} \otimes l_{\gamma\alpha}$ is equivalent to a trivial line bundle on $U_\alpha \cap U_\beta \cap U_\gamma$

3. trivialization $\lambda_{\alpha\beta\gamma}$ satisfies

$$\delta\lambda = 1 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$$

NC gerbes

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U_α some covering

$\star_\alpha$ deformation of θ_α

on $U_\alpha \cap U_\beta \equiv U_{\alpha\beta}$

$$\theta_\alpha = \theta_\beta (1 + \hbar F_{\beta\alpha} \theta_\beta)^{-1}$$

$F_{\beta\alpha}$ - curvature of $\ell_{\beta\alpha} \in Pic(U_{\alpha\beta})$

$U^i_{\alpha\beta}$ - good covering of $U_{\alpha\beta}$

$+ \quad L_{\beta\alpha} = \{ G^{ij}_{\alpha\beta}, \mathcal{D}^i_{\alpha\beta}, \star_\alpha \}$

$$G^{ij}_{\alpha\beta} \star_\alpha G^{jk}_{\alpha\beta} = G^{ik}_{\alpha\beta} \qquad G^{ii}_{\alpha\beta} = 1$$

$$\mathcal{D}^i_{\alpha\beta}(f) \star_\alpha G^{ij}_{\beta\alpha} = G^{ij}_{\alpha\beta} \star_\alpha \mathcal{D}^j_{\alpha\beta}(f)$$

$$\mathcal{D}^i_{\alpha\beta}(f \star_\beta g) = \mathcal{D}^i_{\alpha\beta}(f) \star_\alpha \mathcal{D}^i_{\alpha\beta}(g)$$

Axioms

1. $L_{\alpha\beta} = \{ G^{ij}_{\beta\alpha}, D^{i}_{\beta\alpha}, *_\beta \}$ and

$L_{\beta\alpha} = \{ G^{ij}_{\alpha\beta}, D^{i}_{\alpha\beta}, *_\alpha \}$ are related

as

$$G^{ij}_{\beta\alpha} = D^{j\,-1}_{\alpha\beta}\left(G^{ji}_{\alpha\beta}\right)$$

$$D^{i}_{\beta\alpha} = D^{i\,-1}_{\alpha\beta}$$

$$L_{\alpha\beta} = L^{-1}_{\beta\alpha}$$

2. On $U_\alpha \cap U_\beta \cap U_\gamma$

$$L_{\alpha\beta} \otimes L_{\beta\gamma} \sim L_{\gamma\alpha}$$

$$G^{ij}_{\alpha\beta} *_\alpha D^{j}_{\alpha\beta}\left(G^{ij}_{\beta\gamma}\right) = \Lambda^{i}_{\alpha\beta\gamma} *_\alpha G^{ij}_{\alpha\gamma} * \left(\Lambda^{j}_{[illegible]}\right)^{-1}$$

$$D^{i}_{\alpha\beta} \circ D^{i}_{\beta\gamma} = Ad_{*_\alpha} \Lambda^{i}_{\alpha\beta\gamma} \circ D^{i}_{\alpha\gamma}$$

3. On $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

$$\Lambda^{i}_{\alpha\beta\gamma} *_\alpha \Lambda^{i}_{\alpha\gamma\delta} = D^{i}_{\alpha\beta}\left(\Lambda^{i}_{\beta\gamma\delta}\right) *_\alpha \Lambda^{i}_{\alpha\beta\delta} =$$

$$\Lambda^{i}_{\alpha\beta\gamma} = \left(\Lambda^{i}_{\alpha\gamma\delta}\right)^{-1} \qquad \Lambda^{i}_{\alpha\beta\gamma} = D^{i}_{\alpha\beta}\left(\Lambda^{i}_{\beta[illegible]}\right)$$

Axions are consistent

Equivalence – All NC line bundles replaced by equivalent ones

Continuous connection on gerbe:

$D_{\alpha\beta}$ connection on $L_{\beta\alpha}$

$D_{\alpha\beta\gamma}$ connection on $L_{\alpha\gamma} \otimes L_{\gamma\beta} \otimes L_{\beta\alpha}$

$$D^i_{\alpha\beta\gamma} \Lambda^i_{\alpha\beta\gamma} = 0$$

---

let now $F_{\alpha\beta} = da_{\alpha\beta}$ on $U_\alpha \cap U_\beta$

$\Rightarrow$ $L_{\beta\alpha}$ are trivial

$$G^{(ij)}_{\alpha\beta} = {H^i_{\alpha\beta}}^{-1} \star_\alpha H^j_{\alpha\beta}$$

$$D_{\alpha\beta} = Ad_{\star_\alpha} H^i_{\alpha\beta} \circ D^i_{\alpha\beta}$$

$$\Lambda_{\alpha\beta\gamma} = H^i_{\alpha\beta} \star_\alpha D^i_{\alpha\beta}(H^i_{\beta\gamma}) \star_\alpha D^i_{\alpha\beta} D^i_{\beta\gamma}(H^i_{\gamma\alpha}) \star_\alpha \star_\alpha \Lambda^i_{\alpha\beta\gamma}$$

is globally defined on $U_\alpha \cap U_\beta \cap U_\gamma$

+ on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

$$\Lambda_{\alpha\beta\gamma} *_\alpha \Lambda_{\alpha\gamma\delta} = D_{\alpha\beta}(\Lambda_{\beta\gamma\delta}) *_\alpha \Lambda_{\alpha\beta\delta}$$

$$\Lambda_{\alpha\beta\gamma} = \Lambda_{\alpha\gamma\beta}^{-1} \qquad D_{\alpha\beta}(\Lambda_{\beta\gamma\alpha}) = \Lambda_{\alpha\beta\gamma}$$

$$D_{\alpha\beta} \circ D_{\beta\gamma} \circ D_{\gamma\alpha} = Ad_{*_\alpha} \Lambda_{\alpha\beta\gamma}$$

$\Downarrow$

Definition for good coverings

$(\Lambda_{\alpha\beta\gamma}, D_{\alpha\beta})$ – trivial if there is a global $*$ on M + collection of "twisted" transition functions $G_{\alpha\beta}$ + local equivalences D_α of $*$-products

$$D_\alpha(f) * D_\alpha(g) = D_\alpha(f *_\alpha g)$$

$$G_{\alpha\beta} * G_{\beta\gamma} = D_\alpha(\Lambda_{\alpha\beta\gamma}) * G_{\alpha\gamma}$$

and $\quad Ad_* G_{\alpha\beta} \circ D_\beta = D_\alpha \circ D_{\alpha\beta}$

$\mathcal{A}_\alpha \equiv D_\alpha - id$

$\mathcal{A}_{\alpha\beta} \equiv D_{\alpha\beta} - id$

twisted gauge transformations

$$\mathcal{A}_\alpha = Ad_* G_{\alpha\beta} \circ \mathcal{A}_\beta + G_{\alpha\beta} * d(G_{\alpha\beta})^{-1} - D_\alpha \circ \mathcal{A}_{\alpha\beta}$$

Quantization of twisted Poisson structures (Weinstein + Ševera)

$H \in H^3(M, \mathbb{Z})$ closed, integral 3-form

U_α - good covering $\quad H|_{U_\alpha} = dB_\alpha$

on $U_\alpha \cap U_\beta \quad B_\alpha - B_\beta = da_{\alpha\beta}$

on $U_\alpha \cap U_\beta \cap U_\gamma$

$$a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha} = -i\lambda_{\alpha\beta\gamma} d\lambda^{-1}_{\alpha\beta\gamma}$$

$\lambda_{\alpha\beta\gamma}$ - defines a gerbe

let $\quad \theta = \theta^{(0)} + \hbar\theta^{(1)} + \dots$ antisym bivector field

such that

$$[\theta, \theta] = \hbar\theta^* H$$

Poisson twisted by H

$$(\theta^* H)^{ijk} = \theta^{ii_1}\theta^{jj_1}\theta^{kk_1} H_{i_1 j_1 k_1}$$

On each $U_\alpha \quad \theta_\alpha = \theta(1 - \hbar B_\alpha \theta)^{-1}$

$$[\theta_\alpha, \theta_\alpha] \text{ is Poisson}$$

on $U_\alpha \cap U_\beta$

$$\theta_\alpha = \theta_\beta (1 + \hbar F_{\beta\alpha} \theta_\beta)^{-1}$$

with $F_{\beta\alpha} = d a_{\beta\alpha}$

Formality $\theta_\alpha \to *_\alpha$

on $U_{\alpha\beta}$

$$\mathcal{D}_{\alpha\beta}(f) *_\alpha \mathcal{D}_{\alpha\beta}(g) = \\ = \mathcal{D}_{\alpha\beta}(f *_\beta g)$$

$+ \quad \mathcal{D}_{\alpha\beta} \circ \mathcal{D}_{\beta\gamma} \circ \mathcal{D}_{\gamma\alpha} = Ad_{*_\alpha} \Lambda_{\alpha\beta\gamma}$

$\Downarrow$

$\Lambda_{\alpha\beta\gamma}$ – defines a quantum gerbe.

WZW Poisson sigma model

KLIMCIK & STROBL, PARK

$$S = \int_{\Sigma} d^2\sigma \; dX^i \wedge \eta_i + \theta^{ij}(X)\, \eta_i \wedge \eta_j + \int_V H.$$

Σ – cylinder

η_i – one form

X^i – local coordinates on M

constraints are first class if

$$[\theta, \theta] = \theta^* H$$

locally $H|_{U_\alpha} = dB_\alpha$ and constraints can be solved

$$S_\alpha = \int_{\Sigma} d^2\sigma \; dX^i \wedge \eta_i + \theta_\alpha^{ij}(X)\, \eta_i \wedge \eta_j$$

$\star_\alpha$ – Cattaneo & Felder