GRO[illegible]E

# QFT on NC space-time

Kopaonik 2002

## I, Introduction

1) Space - Time concepts

2) Deformed QM

3) Undeformed QFT { a) Minkowski - Operatorf. b, Euclidean c, Pert. exp - Renormaliz.

## II, Regularization

1) Ideas

2, Matrix Geometry 92 Madore

3, Fuzzy Sphere Klimcik

4, Scalar field regul. [illegible]najder

5 top. sectors Madore

6, Supersphere Steinacker

7, Fuzzy q-sphere

## III, Renormalization

1) Introduction

2, Formulation of models Schweda

3, IR/UV mixing Bichl

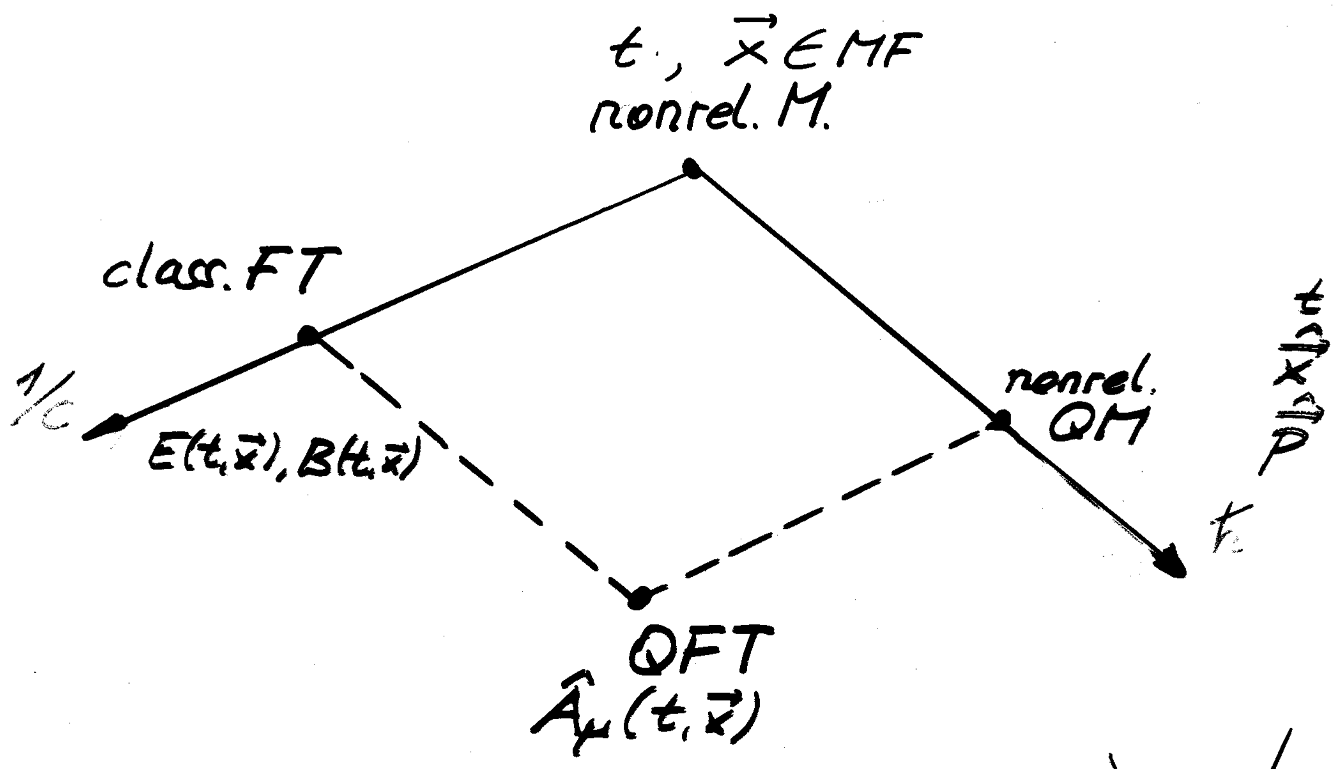
4, Renormalizability ? Grimstrup

5, expand in $\Theta$, use SW? Popp

6, new attempts Wulkenhaar

# I) Introduction

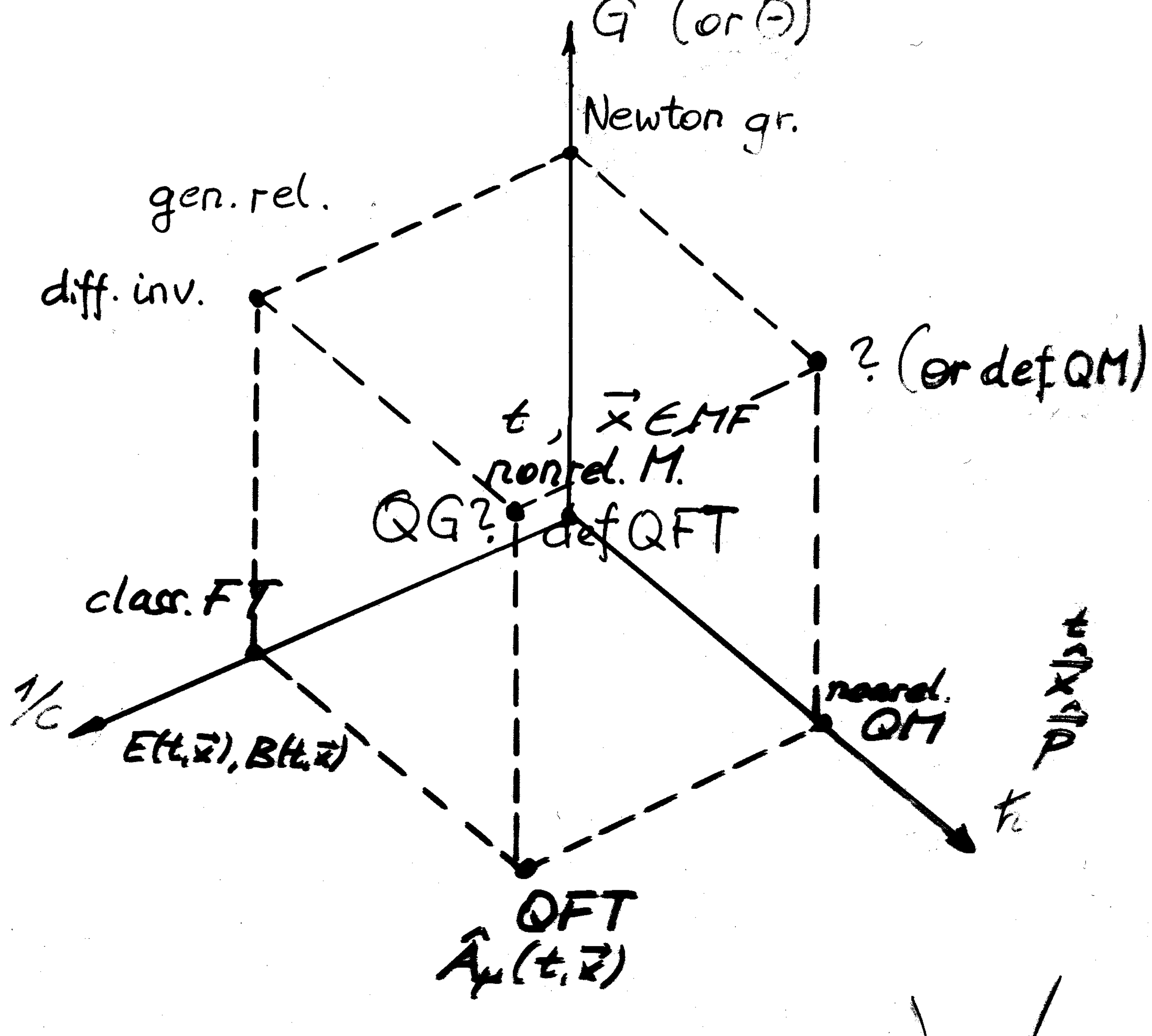
## 1) Space Time concepts




+ Temp.

# I) Introduction

## 1) Space Time concepts




+ Temp.

$$E = mc^2$$

constants $\rightsquigarrow$ scales

$$R_{ke} = \frac{e^2}{mc^2}$$

$$\{\hbar, c, e, m\} \rightsquigarrow \alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137,..}$$

$$\lambda_c = \frac{\hbar}{mc}$$

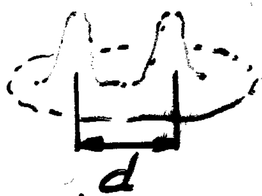
$$\lambda_{Bohr} = \frac{\hbar^2}{e^2 m}$$

localize event $\boxed{E = \frac{\hbar c}{\lambda_{de Broglie}}}$

$$\{\hbar, c, G, m\} \qquad \frac{G m \cdot c^2}{R_{SS}} = \cancel{m} c^2 \cdot c^2$$

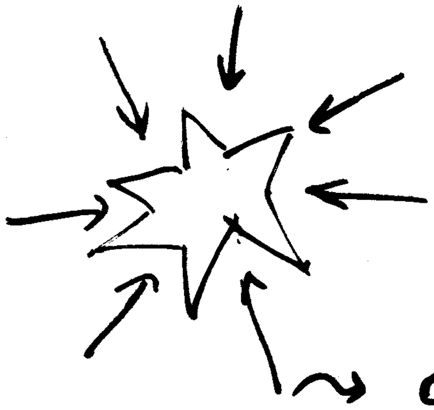
let $\lambda \overset{0}{<} d$

$$\rightsquigarrow R_{SS} = \frac{GE}{c^4} \underset{QM}{=} \frac{G\hbar}{c^3 \lambda} \overset{0}{>} \frac{G\hbar}{c^3 d}$$




**require** $d \overset{0}{>} R_{SS} > \frac{G\hbar}{c^3 d}$

$$\Rightarrow d > \lambda_{Pl} = \sqrt{\frac{G\hbar}{c^3}} \simeq 10^{-35}\,m$$



energy gen. grav. field

strong grav. f. prevents signals to reach obs.

$\rightsquigarrow$ operational impossible to localize events $\leq \lambda_{Pl}$

95 DFR .... ST uncert. rel.

## Über die Unanwendbarkeit der Geometrie im Kleinen.

Von E. Schrödinger, zur Zeit Oxford.

Mit Unanwendbarkeit meine ich: Unzweckmäßigkeit der Anwendung. Mit Anwendung: physikalische Anwendung. Außerdem kann ich meine These nicht beweisen, bloß wahrscheinlich machen. „Über die wahrscheinliche Unzweckmäßigkeit der physikalischen Anwendung der Geometrie im Kleinen" war jedoch als Titelzeile ungeeignet. — Den Anlaß zu dieser Mitteilung bildet von Laues jüngste Note in dieser Zeitschrift[1], der ich, wie auch seiner früheren[2] und seinem Aufsatz in der Scientia[3], aus vollem Herzen zustimme. Der Zweck ist, in dasselbe Horn zu stoßen, damit es mannigfaltiger und lauter ertöne. Und so sollen vorab noch ein paar andere Bemerkungen Platz finden.

Daß man prinzipiell nicht beliebig genau messen kann, wird gern mit der prinzipiell endlichen (nicht unendlich kleinen) Rückwirkung des Probekörpers oder Meßinstrumentes auf das auszumessende Objekt begründet, was von Laue ebenso verführ- übertragung, diese aber erfolge in bestimmten e lichen Quanten, mit denen man nicht zur Gre Null übergehen kann. Man vergißt dabei, wie scheint, daß erstens Wechselwirkung nicht n wendig *Energie*übertragung sein muß; es kann z. einfach nur ein fliegendes Teilchen aus seiner *Ri tung* abgelenkt werden, und zwar um einen belie kleinen Winkel. (Vielleicht ist die Ablenkung Grund der Heisenbergrelation nicht beliebig gen *meßbar*; aber darauf darf man sich doch ni stützen, wenn man eine direkt einleuchtende gründung für die beschränkte Meßgenauigk geben will![1])

Zweitens scheint mir vergessen, daß überha die Energieniveaus eines Systems nicht immer e diskrete Folge mit endlichen Abständen, sond sehr oft auch ein Kontinuum bilden. Dann ist e beliebig kleine Energieübertragung denkbar, ge wie in der Klassik. Es ist jedenfalls von vornher nicht einzusehen, was den Grenzübergang verbie



# Die Grenzen der Anwendbarkeit der bisherigen Quantentheorie.

Von **W. Heisenberg**, Leipzig.

Mit 2 Abbildungen. (Eingegangen am 24. Juni 1938.)

**Stellt man die Quantentheorie der Wellenfelder in einer Form dar, in der ihre relativistische Invarianz besonders einfach erkennbar wird, so bieten sich gewisse natürlich scheinende Annahmen über die Grenzen dar, bis zu denen die Anwendung der bisherigen Quantentheorie gerechtfertigt ist (Abschnitt 1). Diese Annahmen werden auf einige spezielle Fragen angewendet (Abschnitt 2). Schließlich werden die Erscheinungen besprochen, die außerhalb des Anwendungsbereichs der bisherigen Theorie liegen (Abschnitt 3).**

Bekanntlich verbieten die in der Quantentheorie der Wellenfelder auftretenden Divergenzen bisher die Formulierung einer in sich geschlossenen Quantentheorie der Elementarteilchen[1]), die von den in der Kernphysik und der Höhenstrahlung beobachteten Erscheinungen von selbst Rechenschaft geben müßte. Dieser Umstand legt die Vermutung nahe, daß in der Theorie der Elementarteilchen eine universelle Konstante von der Dimension einer Länge eine grundsätzliche Rolle spielt und daß

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# Quantized Space-Time

HARTLAND S. SNYDER
*Department of Physics, Northwestern University, Evanston, Illinois*
(Received May 13, 1946)

It is usually assumed that space-time is a continuum. This assumption is not required by Lorentz invariance. In this paper we give an example of a Lorentz invariant discrete space-time.

THE problem of the interaction of matter and fields has not been satisfactorily solved to this date. The root of the trouble in present field theories seems to lie in the assumption of point interactions between matter and fields. On the other hand, no relativistically invariant Hamiltonian theory is known for any form of interaction other than point interactions.

Even for the case of point interactions the relativistic invariance is of a formal nature only, as the equations for quantized interacting fields have no solutions. The uses of source functions, or of a cut-off in momentum space to replace infinity by a finite number are distasteful arbitrary procedures, and neither process has yet been formulated in a relativistically invariant manner. It may not be possible to do this.

It is possible that the usual four-dimensional continuous space-time does not provide a suitable framework within which interacting matter and fields can be described. I have chosen the idea that a modification of the ordinary concept of space-time may be necessary because the "elementary" particles have fixed masses and associated Compton wave-lengths.

The special theory of relativity may be based on the invariance of the indefinite quadratic form

$$S^2=c^2t^2-x^2-y^2-z^2, \tag{1}$$

a total of forty-five commutators. Only six of these commutators differ from the ordinary ones and these six are

$$\begin{aligned}
[x,y]&=(ia^2/h)L_z, & [t,x]&=(ia^2/hc)M_x,\\
[y,z]&=(ia^2/h)L_x, & [t,y]&=(ia^2/hc)M_y,\\
[z,x]&=(ia^2/h)L_y, & [t,z]&=(ia^2/hc)M_z.
\end{aligned} \tag{5}$$

We see from these commutators that if we take the limit $a\rightarrow 0$ keeping $h$ and $c$ fixed, our quantized space-time changes to the ordinary continuous space-time.

ordinates and time are not the same as usual and are given by

$$\begin{aligned}
&[x,p_x]=ih[1+(a/h)^2px^2];\\
&[t,p_t]=ih[1-(a/hc)^2p_t^2];\\
&[x,p_y]=[y,p_x]=ih(a/h)^2p_xp_y;\\
&[x,p_t]=c^2[p_x,t]=ih(a/h)^2p_xp_t;\ \text{etc.}
\end{aligned} \tag{8}$$

Here we note that if all the components of the momentum are small compared to $h/a$ and the energy is small compared to $hc/a$ then these commutators approach those which are given in

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# The Electromagnetic Field in Quantized Space-Time

HARTLAND S. SNYDER
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(Received February 26, 1947)

Relativistically invariant equations of motion for the electromagnetic field are set up in quantized space-time. These equations are solved by a process similar to a Fourier analysis.

IN a previous paper[1] it was shown that a Lorentz invariant space-time is not necessarily a continuum, and an example was given of a discrete Lorentz invariant space-time. This paper is a report on work done to determine whether relativistically invariant field equations could be introduced into quantized space-time, and whether such field equations are solvable. In continuous space-time the field quantities are taken to be functions of the space and time coordinates. Consequently, I assume that if $A(x,y,z,t)$ is a field quantity of continuous space-time and if a term of the form $\partial A/\partial x$ appears in the field equations, this term will be replaced by $i[p_x, A]$ in the transition to quantized space-time. It is evident that if $A$ is a Hermitian operator, then $i[p_x, A]$ will also be Hermitian, so that this replacement of partial derivatives preserves reality conditions. If we make the replacements suggested above into the usual form of the vacuum Maxwell's

## 2) Ex.: deformed QM

Cl. Mechanics $\xrightarrow{\text{def.}}$ Qu. Mechanics

sympl. MF $(M,\omega)$ $\qquad\qquad$ $\mathcal{H}$

$\omega = dx \wedge dp_1 + \dots$ $\qquad$ $[\hat{x}_i, \hat{p}_j] = i\,\delta_{ij}$

Observables $\qquad\qquad$ or Weyl!

comm. Algebra $\{f(\Gamma)\}$ $\qquad$ noncomm. alg $\mathcal{B}(\mathcal{H})$

time aut.

$H(x,p)$ $\qquad\qquad$ $\hat{H}(\hat{x},\hat{p})$

states

pos. measures $\qquad\qquad$ pos. l. fcts.: $\begin{cases} \langle 0 | a_t | 0 \rangle \\ \mathrm{Tr}\,\rho a \end{cases}$

$\langle F \rangle = \int d\Gamma\, \rho F$

---

**Remarks: def. quant.:**

$$\underbrace{A(x,p)}_{z} = \int dy\, \langle x-y | \hat{A} | x+y \rangle e^{ipy}$$

dequ.

$$\hat{A}(\hat{x},\hat{p}) = \int d\alpha \int d\beta\, e^{i\alpha\hat{x} + i\beta\hat{p}} A(\alpha,\beta)$$

$$\hat{A} \cdot \hat{B} \longrightarrow (A * B)(z) = e^{i\Theta^{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j}} A(x)\, B(y) \Big|_{x=y=z}$$

$$\Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{Moyal-Weyl}$$

---

Weylalgebra: $\underbrace{e^{i\alpha\hat{x}}}_{U_1^{\alpha}}\ \underbrace{e^{i\beta\hat{p}}}_{U_2^{\beta}} = e^{-i\alpha\beta}\, e^{i\beta\hat{p}}\, e^{i\alpha\hat{x}}$

on $S(\mathbb{R})$ $!$ irep.

$$(U_1^{\alpha} f)(x) = f(x+\alpha), \quad (U_2^{\beta} f)(x) = e^{i\beta x} f(x)$$

---

fix $\alpha$ & $\beta$: $\quad U_1\, U_2 = e^{-i\theta}\, U_2\, U_1$

$\{\sum c_{nm} U_1^n U_2^m\}$

$\theta$ irrational $\Rightarrow$ $T_\theta^2$ irr. Torus

Weylrep.

$\theta = \frac{1}{N}$ $\quad\exists$ finite dim rep. in $\mathbb{C}^N$

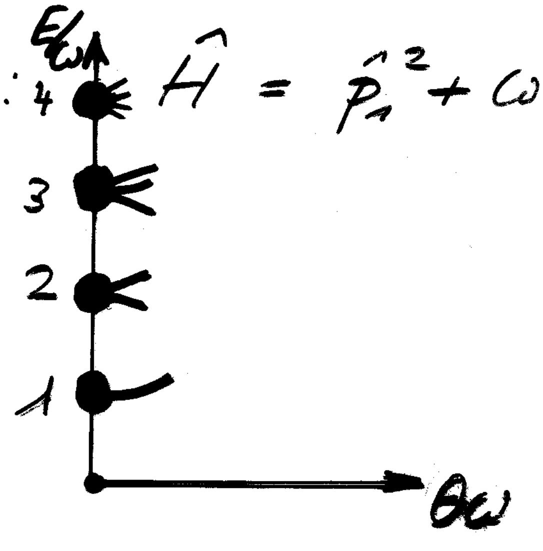
$$U_1 = \begin{pmatrix} 1 & & & \\ & e^{i/N} & & \\ & & \ddots & \\ & & & e^{i\frac{N-1}{N}} \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}$$

---

QM $\quad d=2$, $L^2(\mathbb{R}^2)$

def.: $[\hat{x}_1, \hat{x}_2] = i\theta$, $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$, $[\hat{p}_i, \hat{p}_j] = 0$

Osc.: $\hat{H} = \hat{p}_1^2 + \omega^2 \hat{x}_1^2 + \hat{p}_2^2 + \omega^2 \hat{x}_2^2$

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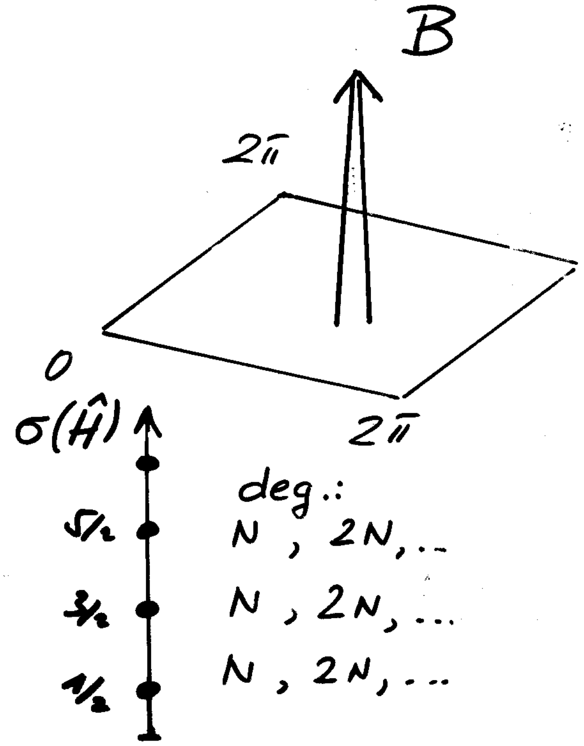
LANDAU :

$$\mathcal{H} = L^2(\mathbb{T}^2)$$




$$\hat{H} = (\vec{\hat{p}} + \vec{\hat{A}})^2 = -\nabla_1^2 - \nabla_2^2$$

$$\text{gauge} \quad A = \begin{cases} \frac{B}{2}\hat{y} \\ -\frac{B}{2}\hat{x} \end{cases}$$

Fluxquant.:

$$(2\pi)^2 B = 2\pi \frac{N}{n} \leftarrow \text{supercell}$$

gauge transf.: $g = e^{i\alpha(x_1, x_2)}$ / b.c.

$$\nabla \to g \nabla g^{-1}$$

---

deform: $[\hat{x}_1, \hat{x}_2] = i\theta \,, \quad [\hat{x}_i, \hat{p}_j] = i\delta_{ij} \,, \quad [\hat{p}_i, \hat{p}_j] = 0$

cov. coord. $\quad \hat{x}_i + \theta \epsilon_{ij} \hat{A}_j = \hat{\xi}_i \longrightarrow g \hat{\xi}_i g^{-1}$

rescale $\quad [\xi_1, \xi_2] = i\theta \,, \quad [\xi_i, i\nabla_j] = i\delta_{ij} \,, \quad [i\nabla_i, i\nabla_j] = i\epsilon_{ij} B$

$$\xi \rightsquigarrow U_j = e^{i\xi_j} \quad \to \quad \mathbb{T}^2_\Theta : \quad U_1 U_2 = e^{-i\Theta} U_2 U_1$$

Standard rep.: $\quad U_j^{\mathcal{H}} = e^{-\frac{1}{B}\nabla_j}$

Write $U_j = U_j^{st} \cdot u_j$

since
$U_1^{st} U_2^{st} = e^{+iB^{-1}} U_2^{st} U_1^{st}$

$\Rightarrow u_1 u_2 = e^{-in/N} u_j u_i$

cyclic group

represent $U_j$ in $S(\mathbb{R} \times \mathbb{Z}_N)$

qu.cond.: $\Theta = -\frac{1}{B} + \frac{n}{N}$ Rieffel ...

repr.: $(U_1 \phi)_j(x) = \phi_{j-1}(x - \frac{n}{N} + \Theta)$, $(U_2\phi)_j(x) = \phi_j(x) e^{i(x - j\frac{n}{N})}$

if $n - N\Theta \neq 0$ give projective modules

---

analog: vector space over $\mathbb{C}$

for algebra $\mathcal{A} \longrightarrow \mathcal{A}^N$: columns $\begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} =$ free modul $= a_1 e^1 + \dots + a_N e^N$

more general:
proj. modul $E$: $\mathcal{A}^N \xrightarrow{P} P\mathcal{A}^N$

# 3) Undeformed QFT

## a) Minkowski, Operatorform.:

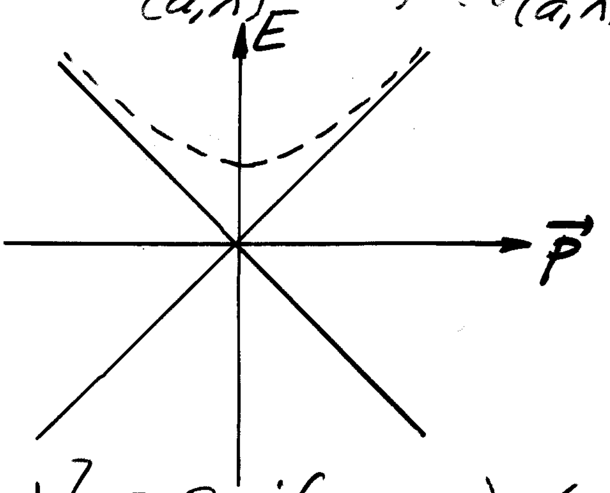
Fockspace $\mathcal{F} = \bigoplus_n (\mathcal{H} \wedge \ldots \wedge \mathcal{H})_n$

Fields: $\phi(x) \rightsquigarrow \phi(f) = \int dx\, f^*(x)\, \phi(x)$

Covariance: $U_{(a,\Lambda)}\, \phi(f)\, U^+_{(a,\Lambda)} = \phi(f_{(a,\Lambda)})$

Spectrum cond.

$\sigma(H,P) \subset \overline{V_+}$:




Locality:

causality: $[\phi(f), \phi(g)] = 0$ if $\mathrm{sup}f \times \mathrm{sup}g$

$\exists!$ vacuum

$\exists$ reconstruction th..:

$\{\langle 0|\, \phi(f_1) \ldots \phi(f_N)\, |0\rangle\} \xrightarrow[\text{recover}]{} $ Theory

Like GNS construction

b) Eucl. QFT

analyt. cont

QM $\to$ diff. equ

$$e^{-\tau H}(x,y) = \int\limits_{\substack{0\ldots\tau \\ x\ldots y}} d\mu(\omega)\, e^{-\int_0^\tau d\tau V(\omega(\tau))} = \int [d\omega]\, e^{-S_E[\omega]}$$

N pt Wightmanfct.:

$$S_N(x_1 \ldots x_N) = \langle 0 | \phi(it_1, \vec{x}_1) \ldots \phi(it_N, \vec{x}_N) | 0 \rangle$$

Axioms....

esp.: Reflection / OS } positivity: $\langle \varphi(\underbrace{f}_{f(\tau,\vec{x})}) \varphi(\underbrace{\bar{\Theta} f}_{f^*(-\tau,\vec{x})}) \rangle \geq 0$

connect to stat. ph.:

canonical ens.: $\langle F \rangle = \dfrac{\mathrm{Tr}\, e^{-\beta H} F}{\underbrace{\mathrm{Tr}\, e^{-\beta H}}_{Z}}$ $Z$ partition fct.

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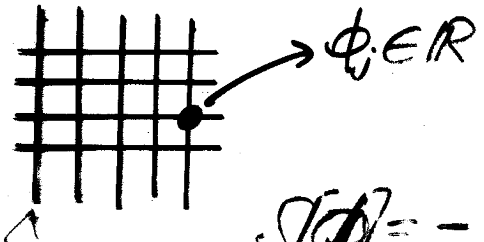
Ex.: Lattice regul.:




$$Z[J] = \int \prod_{j \in \Lambda} d\phi_j\, e^{-S[\phi]}$$

$$S[\phi] = -\sum_j (\phi_j - \phi_{j+a})^2 + \sum_j \phi_j J_j + \sum_j V(\phi_j)$$

destroys symmetries ; gauge fields alt.

---

c) Feynman perturbation expansion

Magic formula of Gellmann & Law

$$\langle 0 | T(\phi_H(x_1)\dots\phi_H(x_N)) | 0\rangle =$$

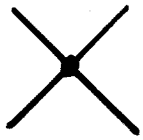
$$= \frac{\langle 0 | T(\varphi(x_1)\dots\varphi(x_N)\, e^{i\int \mathcal{L}_I(\varphi)}) | 0\rangle}{\langle 0 | T(e^{i\int \mathcal{L}_I(\varphi)}) | 0\rangle}$$

expand connected part of $N$ pt fct:

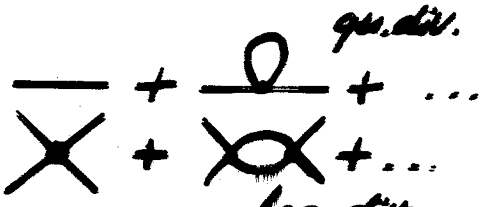
**p-th order:**

$$\tau^N(x_1\dots x_N)^{(p)} = \frac{i^p}{p!\,N} \int_{y_1\dots y_p} \langle 0 | T(\varphi(x_1)\dots\varphi(x_N) :\mathcal{L}_I(y_1):\dots :\mathcal{L}_I(y_p):) | 0\rangle$$

Wick: contract → F. rules

Ex: $\phi^4$   verticer    $\langle 0|T(\varphi\varphi)|0\rangle$ prop —— internal / external


Ex.: $N=2$ / $N=4$




3 problems:

UV-div. $(|p| < \Lambda, \Lambda \to \infty)$   renormalize

IR-div $(m \to 0, Vol \to \infty)$   finite number of constants

Convergence of pert. exp.   in

$$\mathcal{L}(\varphi) = c_1\, \partial\phi\partial\phi + c_2 \phi^2 + c_3 \phi^4$$

impose normal. cond.: $\Gamma_2(p)\big|_{p^2=m^2} = 0$, $\frac{\partial \Gamma_2(p)}{\partial p^2}\big|_{p^2=m^2} = 1$, $\Gamma_4(p)\big|_{p^2} = c$;

# Status QFT

Dimension :

1 QM ✓

2 models ✓ — constructed, solved: YB, qu.gr, ...

3 few models ✓

4 ? gen. free fields

renorm. pert. th.

em ∪ weak ∪ strong

SUSY SUGRA

STRINGS 10d

ways out : couple many fields

or

NCG – new space time concept, generalize geometry

apply to models

steps :

| Manifold | nc Algebra |
|---|---|
| diff. calculus | |
| vector f. | derivations |
| differentials ... | dual to deriv. |
| fields | |
| scalar<br>spinor<br>vector | fin. generated<br>proj. modules |
| integration | |
| $\int_c \omega$ | cyclic cohomology |

Spectral triple ???

$(\mathcal{A}, \mathcal{H}, D)$

1, Ideas:

encode structure in comm. alg. and deform

= quantize

Ex.: QM $\{q,p\} \rightarrow [\hat{q},\hat{p}] \neq 0$ cells !

+ magn. field $[\Pi_x, \Pi_y] \propto B_z \neq 0$

Landau cells

---

ST:

$$[x^\mu, x^\nu] = \begin{cases} -2i\Theta^{\mu\nu} & \text{canonical} \leftarrow \\ i\,\epsilon^{\mu\nu\rho} x_\rho & \text{Lie algebra} \\ \tilde{R}^{\mu\nu}{}_{\rho\sigma} x^\rho x^\sigma & \text{Quantum group} \end{cases} \quad \text{def.}$$

---

Use differential calculus

Vector fields → derivations

differential f → duals

define fields

scalar, spinor, vector.. → fin. gen. proj. modules

Integration, coh.... → spectral triples

cyclic coh. ...

spectral action principle....

## 2, Matrix Geometry - Diff. calculus on $M_n$

Algebra: $\lambda_i$ base Lie $SU(n)$, $i=1,\dots n^2-1$

$$\lambda_i \lambda_j = \frac{c_{ijk}}{2} \lambda_k + \frac{d_{ijk}}{2} \lambda_k - \frac{1}{n} g_{ij}$$

$\{1, \lambda_i\}$ generates $M_n$

---

Vector Fields: adj. action of Liealg.

Derivations: $e_i = \mathrm{ad}\, \lambda_i$ $n^2-1$ l.i. v.f.

Ex.: $e_i(\lambda_j) = [\lambda_i, \lambda_j] = c_{ijk} \lambda_k$

$$[e_i, e_j] = c_{ijk} e_k$$

v.f. do not form a left modul over $M_n$

---

**Diff. forms:**

$\Omega^0(M_n) = M_n$ $\Omega^1(M_n) = \{ a db \mid a, b \in M_n \}$

$d(1) = 0$, $\boxed{(df)(e_i) = e_i(f) 1}$

Ex $(d\lambda_j)(e_i) = e_i(\lambda_j) = c_{ijk} \lambda_k$

dual basis $\Theta^i(e_j) = \delta^i_j$ $\quad d\lambda^e = c^{emn}\lambda_m\Theta_n$

$\Omega^1$ is left modul over $M_n$

$\Theta^i \wedge \Theta^j = -\Theta^j \wedge \Theta^i, \quad \lambda^i\Theta^j = \Theta^j\lambda^i, \ldots$

extend $\Omega^1 \xrightarrow{d} \Omega^2$

$$(d\Theta^k)(e_i, e_j) = \frac{1}{2}\left\{e_i(\Theta^k(e_j)) - e_j(\Theta^k(e_i)) - \Theta^k([e_i, e_j])\right\}$$

Maurer Cartan equ. $\quad = -\frac{1}{2} c_{emk}\, \Theta^e \wedge \Theta^m (e_i, e_j)$

Leibniz ...

$\exists$ special form $\quad \Theta = -\sum \lambda_i \Theta^i$

$d\Theta + \Theta \wedge \Theta = 0$ $\quad$ "M C form"

$\Rightarrow \quad df = (e_i f)\,\Theta^i = -[\Theta, f]$ on $M_n$

"Dirac op."

---

Cov. der., gauge tr., Lie deriv., $*$, $\Delta$, ...

Cov. der. $\not{D} = d + A$

$$A = a_i \Theta^i$$

$$D^2 = dA + A^2 = F = \frac{F_{ij}}{2} \Theta_i \wedge \Theta_j$$

Gauge transf. $g \in SU(n)$

$$A \to g^{-1} A g + g^{-1} dg,$$

$$F \to g^{-1} F g \quad , \quad \Theta \to \Theta$$

---

$$S[A] = -\frac{1}{4} \, \mathrm{Tr} \, F_{ij} F^{ij}$$ g.i.

---

Vol.f.: $v = \sqrt{|g|} \, \Theta^1 \wedge \ldots \wedge \Theta^{n^2-1}$ , $*$

$\delta = *^{-1} d *$ , $\delta d + d \delta = \Delta$ L.B.Op.

.....

Lie deriv. $\mathcal{L}_X \omega \equiv (i_X \circ d + d \circ i_X) \, \omega$ , ...

---

# 3, Fuzzy Sphere

Undeformed sphere $S^2 \hookrightarrow \mathbb{R}^3$, $\quad x^i$, $[x_i, x_j] = 0$

$i = 1, 2, 3$

$\mathcal{A}_\infty = \{ f(x^i) \mid \text{anal.} \} / I_\infty$ ,

$\nwarrow$ fct. v. at $x^2 = \rho^2$

$$\langle f_1 | f_2 \rangle = \int \frac{d^3x}{2\pi\rho} \, \delta(x^2 - \rho^2) \, f_1^*(x) f_2(x)$$

gen. of rot.: $J_i = \epsilon_{ijk} x_j \frac{1}{i} \frac{\partial}{\partial x_k}$

rot. inv. action: $S = \langle J_i f | J_i f \rangle$

generator $x_i$ form spin 1 irep. of $SU(2)$

higher powers $\rightarrow$ higher spin reps

$$\mathcal{A}_\infty = [0] \oplus [1] \oplus [2] \oplus \ldots$$

# Regularization

## Truncation!

$$\mathcal{A}_j = [0] \oplus [1] \oplus \dots \oplus [j]$$

Consider maps from $[\frac{j}{2}] \longrightarrow [\frac{j}{2}] \equiv \mathcal{A}_j$

assoc. product given ?

$\exists$ scalar product

1st order $\mathbb{1}$

2nd order $\mathbb{1}_2, \vec{\sigma}$

$\vdots$

n-th order $\mathbb{1}_n, \vec{J}_n$

$$X^i_{(n)} = \varkappa_n J^i_{(n)}$$

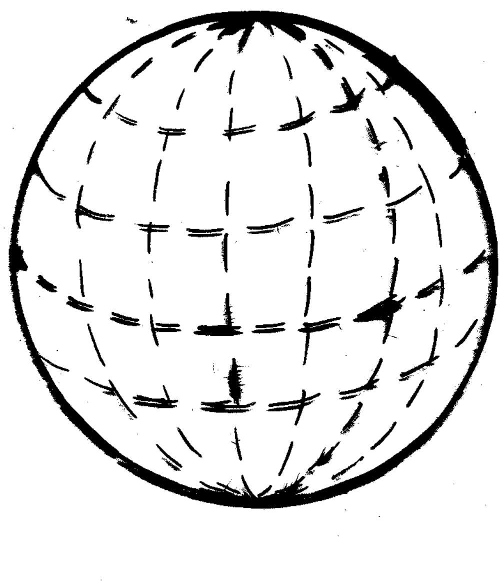
n-th irrep. of $SU(2)$

$$\sum_{i=1,2,3} (X^i_{(n)})^2 = R^2$$

$\exists$ convergence

$$(n^2-1)\varkappa_n^2 = R^2$$

$$\varkappa_n \sim \frac{R}{n}$$



$$[X^i_{(n)}, X^j_{(n)}] = \frac{i\epsilon^{ijk}R}{\sqrt{n^2-1}} X^k_{(n)}$$

Sphere has $n^2$ cells

embeddings, convergence, ... or: use coherent ~~states~~ ~~product~~

Fuzzy Sphere !

$$X_j^{(N)} : \quad \sum_j^3 (X_j^{(N)})^2 = \rho^2$$

$$[X_i^{(N)}, X_j^{(N)}] = \frac{i\,\epsilon_{ijk}\,\rho}{\sqrt{N^2-1}} X_k^{(N)}$$

---

Or: Use Coherent States over $G = SU(2)$

$T_n(g)$ unit irep. in $\mathcal{H}_n$, $|x_0\rangle \in \mathcal{H}_n$

orbit $\quad T_n(g)|x_0\rangle$

stab. group $\quad H = U(1)$, $\quad x \in M = G/H \simeq S^2$

$$|x\rangle = T_n(g_x)\,|x_0\rangle$$

---

$\hat{f} = \int dg\, \tilde{f}(g)\, T_n(g)$ $\qquad \tilde{f}(x) = \langle x | \hat{f} | x \rangle$

op. in $\mathcal{H}_n$

$*$ pr. $\qquad (\tilde{f}_1 * \tilde{f}_2)(x) = \langle x | \hat{f}_1 \hat{f}_2 | x \rangle$

coord. fct. $\Rightarrow$ alg. before ....

---

# 4; Scalar Field Theory - NC-cutoff

$$S_N[\phi,\phi^+] = \frac{1}{N}\,\mathrm{Tr}_N\left\{J_i\phi^+ J_i\phi + \mathrm{Pol}(\phi,\phi^+)\right\}$$

$$J_i \cdot := [x^i, \cdot\,]$$

expand $\phi(x) = \sum_{l=0}^{N}\sum_{m=-l}^{l} a_{lm}\,\psi_{lm}(x)$ $\qquad M_N \downarrow$

$$J_i^2\,\psi_{lm} = l(l+1)\,\psi_{lm}$$

$$D\phi_N D\phi_N^+ = \prod_{lm}^{N} da_{lm}\, da^+_{lm}$$

$$\langle F[\phi,\phi^+]\rangle_N = \frac{1}{Z_N}\int D\phi_N D\phi_N^+\, e^{-S_N[\phi,\phi^+]}\, F[\phi,\phi^+]$$

UV-finite , rot. inv. CUT OFF

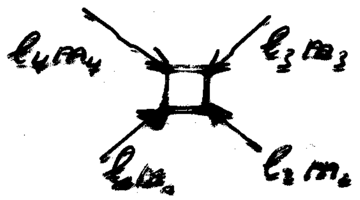
OS-axioms o.k. $\qquad$ preserves symmetries ?

Feynman rules :

ext. lines $\quad \xrightarrow{\;lm\;}$

propagator $\quad l'm' \longrightarrow lm \qquad \dfrac{\delta_{l'l}\,\delta_{m'm}}{(l+\frac{1}{2})^2}$

vertex




, ....

tadpole ~ ln N

5) top. sectors — 2nd method

comm. case:

Hopf fibr.:

$$S^3 \leftarrow U(1)$$
$$\downarrow$$
$$S^2$$

$$S^3_{\sqrt{R}} \qquad\qquad S^2_R$$

$$\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \qquad |\chi_1|^2+|\chi_2|^2 = R \qquad x_i = \chi^+\sigma_i\chi \qquad \sum_i x_i^{\,2} = R^2$$

$$\chi_1 = \sqrt{R}\cos\tfrac{\vartheta}{2}\, e^{-\frac{i}{2}(\varphi+\psi)}$$
$$\chi_2 = -\sqrt{R}\sin\tfrac{\vartheta}{2}\, e^{\frac{i}{2}(\varphi-\psi)}$$

transf

$$\chi_\alpha \to e^{-\frac{i}{2}\psi}\chi_\alpha$$
$$\chi_\alpha^+ \to e^{\frac{i}{2}\psi}\chi_\alpha^+$$

$$\tilde{\mathcal{A}} = \left\{ f = \sum_{\underline{n},\underline{m}} c_{\underline{n}\,\underline{m}}\, \chi_1^{+n_1}\chi_2^{+n_2}\chi_1^{m_1}\chi_2^{m_2} \right\} = \bigoplus_{-\infty}^{\infty} \mathcal{A}_k$$

$$2k = n_1+n_2-m_1-m_2$$

Quantization: Jordan Schwinger real. of $su(2)$

→ n.c. version of Hopf fibration

$$\chi_\alpha \to \sqrt{\tfrac{\lambda}{2}}\, A_\alpha \qquad \text{CCR}$$

$$|n_1,n_2\rangle = (A_1^+)^{n_1}(A_2^+)^{n_2}|0\rangle \qquad \text{BOSON}$$

restrict to irr. subspace $\mathcal{F}_N = \{|n_1,n_2\rangle \mid n_1+n_2 = N\}$

Casimir $C| = \frac{N}{2}\left(\frac{N}{2}+1\right)\ldots$

describe topol. ~~nontr.~~ config.

Let $\hat{A}_{MN}$ lin map. from $\mathcal{F}_N \longrightarrow \mathcal{F}_M$

is left $\hat{A}_M$ & right $\hat{A}_N$ modul

action $\quad J_R f = X_R^{(M)} f - f X_R^{(N)}$

$\rightarrow$ su(2) rep.

$$\left[\frac{M}{2}\right] \otimes \left[\frac{N}{2}\right] = [|k|] \oplus \ \dots \ \oplus \left[\frac{N+M}{2}\right]$$

$$\vec{J}^2 \psi^k_{lm} = l(l+1)\, \psi^k_{lm} \quad , \ l = |k|, \dots \frac{N+M}{2}$$

---

Action $\quad S_{MN}[\phi^+, \phi] = \mathcal{J}_N \left( \phi^+ \frac{JJ^+ + J^+ J}{2} \phi + V(\phi^+, \phi) \right)$

$$J\phi = i A^+_\alpha \epsilon_{\alpha\beta} [A^+_\beta, \phi] \quad , \ J^+ \dots$$

Prop.: $\quad$ SU(2) symm. $\hat{=}$ $S^2$ rot.

finite # of modes

let $\lambda^{-2} = \frac{M+N}{2} \left( \frac{M+N}{2} + 1 \right)$ , $2k = M - N$ fix

$N \to \infty$ $\quad$ scalar f.

---

## 6) Supersphere

Spin 1/2 , Dirac operator , ...

comm. case : spinor bundle $\begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$

$$\mathcal{S} = [\tfrac{1}{2}] \otimes \mathcal{A} = 2 \cdot \left( [\tfrac{1}{2}] \oplus [\tfrac{3}{2}] \oplus \dots \right)$$

Dirac op.: $\qquad \vec{J} = \vec{L} + \frac{\vec{\sigma}}{2}$

$$\mathcal{D}_k = \frac{1}{\rho} \left\{ \vec{\sigma} \cdot \vec{J} + 1 + \frac{k}{2} \vec{\sigma} \cdot \vec{x} \right\}$$

ev.: $\pm (\ell + 1)$ , $m = \ell + \frac{1}{2}, \dots, -(\ell + \frac{1}{2})$

---

Quantization : use superspace

$\mathcal{N} = 1$ ext. of Hopf fibration $\mathbb{C}^{2|1} \longrightarrow \mathbb{R}^{3|2}$

$$f = \begin{pmatrix} \chi_1 \\ \chi_2 \\ a \end{pmatrix} \longrightarrow (x_i, \theta_\mu) \qquad \begin{array}{l} x_i = f^+ R_i f \\ \theta_\pm = f^+ F_\pm f \end{array}$$

odd

$$F_+ = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right) \qquad F_- = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & -1 \\ \hline 1 & 0 & 0 \end{array}\right)$$

$$[F_+, F_-] = -R_3 \quad , \quad [F_\pm, F_\pm] = \pm R_\pm$$

$\Rightarrow osp(2/1)$ superalgebra

---

super $S^2$: $X_i^2 + \frac{1}{2}[\Theta_+, \Theta_-] = \rho^2$

epr. content:

$$s\mathcal{A}_0 = \tilde{0} \oplus \widetilde{1/2} \oplus \tilde{1} \oplus \widetilde{3/2} \oplus \ldots$$

$$\tilde{0} \equiv [0] \quad , \quad \tilde{j} \equiv [j] \oplus [j - \tfrac{1}{2}]$$

---

Truncation:

$$\tilde{0} \oplus \tilde{\tfrac{1}{2}} \oplus \tilde{1} \oplus \ldots \oplus \tilde{j} = [0] \oplus ([0] \oplus [\tfrac{1}{2}]) \oplus ([\tfrac{1}{2}] \oplus [1]) \oplus \ldots$$

$$1 + 1 + 2 + 2 + 3$$

spinor field

$$\psi = f(x^+, x)\, a + g(x^+, x)\, a^+$$

Dirac op.: $D\psi = (J^+ g)\, a + (J f)\, a^+$

grading $\Gamma\psi = f a - g a^+$ chirality

$$\{D, \psi\} = 0$$

Action: $S[\psi^+, \psi] = s\mathrm{Tr}\,(\psi^+ D\psi + W(\psi^+, \psi))$

Spektrum $\qquad E_{kj} = \pm\sqrt{(j+\frac{1}{2})^2 - \frac{k^2}{4}}$ top. nontr. conf.

N×M matrices

$j = \frac{1}{2}(|k|-1), \frac{1}{2}(|k|+1), \ldots, \frac{1}{2}(M+N-1) \qquad k \neq 0$

$\left.\begin{array}{llll} k > 0 & \text{chirality } -1 & \text{sol. } g \\ k < 0 & \text{chirality } +1 & \text{sol. } f \end{array}\right\}$ zero modes

**Results:**

sharp cut off

admissable values of $j$ truncated

couple to gauge field

\# modes finite

SU(2) space symmetry

local gauge invariance

global chiral invariance

UV + IR CUT OFF

top. nontrivial conf. included

Problems: $N \to \infty$, renormalization, anomaly, ...

H.G., Madore, Steinacker

7, Fuzzy q-sphere $\quad q \in \mathbb{R}$ or $|q| = 1$

covariant $\mathcal{U} \equiv U_q(su(2))$

Call $\mathcal{A}$ an $\mathcal{U}$ module alg. if $\quad \mathcal{U} \times \mathcal{A} \to \mathcal{A}$

$$(u, a) \mapsto u \triangleright a \ni u \triangleright (ab) = (u_{(1)} \triangleright a)(u_{(2)} \triangleright b)$$

---

q-def. op.: $A^{+\alpha} A_\beta = \delta^\alpha_\beta + q \hat{R}^{\alpha\gamma}_{\beta\delta} A_\gamma A^{+\delta}$

$$(P^- AA = 0, \ldots)$$

Fockspace $\mathcal{F}_N = \{\sum A^{+\alpha_1} \ldots A^{+\alpha_N} |0\rangle\} \equiv (N+1)$

Def.: $X_i = A^{+\alpha} \epsilon_{\beta\alpha} \sigma_i^{\beta\gamma} A_\gamma \quad , \quad N = A^{+\alpha} \epsilon_{\beta\alpha} \epsilon^{\beta\gamma} A_\gamma$

$$\Rightarrow \epsilon_k^{ij} X_i X_j = (\alpha_q + \beta_q N) X_k$$

$$g^{ij} X_i X_j = N \qquad \text{on } \mathcal{F}_N$$

Define: $S^2_{q,N} = (1) \oplus (3) \oplus \ldots \oplus (2N+1)$

Reality structure ...

Invariant integral $\int \cdot = \mathrm{Tr}_q \cdot$ under qu. adj. act.

$\swarrow$ + scalar f.

$\exists !$ 3 dim
Diff. calculus $\quad S^2_{q,N} \xrightarrow{d} \Omega^1_{q,N}$

free bimodule over $S^2_{q,N}$

Def. one forms $\xi$: $\quad x_i \xi_j = \hat{R}_{ij}{}^{kl} \xi_k x_l$

$\Rightarrow \Theta = x \cdot \xi \quad \Rightarrow \quad df = [\Theta, f]$

$\xi_i \xi_j = -q^2 \hat{R}_{ij}{}^{kl} \xi_k \xi_l$

Def.: Hodge star $\quad * \Theta = \Theta^2 \quad , * 1 = \Theta^3$

$\Omega^1_{q,N} \xrightarrow{d} \Omega^2_{q,N} \quad$ since $\; d\Theta = \Theta^2 \neq 0$

$\Rightarrow [\Theta^2, f] = * df$

$\alpha \overset{d}{\longmapsto} [\Theta, \alpha]_+ - * \alpha \quad \Rightarrow \quad (d \circ d) f = 0$

$\Omega^2_{q,N} \xrightarrow{d} \Omega^3_{q,N} \quad \underline{\alpha \mapsto [\Theta, \alpha]}$

Integration of forms : Gauge fields

$B \in \Omega^1_{q,N} \quad , \quad B = \Theta + A$

$S_2 = \int B * B \qquad S_{CS} = \int (A\,dA + \frac{2}{3} A^3 - \Theta^3/3)$

$S_3 = \int B^3 \quad \longrightarrow \quad$ linear combination

$S_4 = \int B^2 * B^2 \qquad S_{YM} = \int (dA + A^2) * (dA + A^2)$

Alekseev, Recknagel, Schomerus

1) Introduction

2) Regularization

3) Renormalization

I) Introduction — hopes

II) Formulation of models — easy

III) IR/UV mixing — surprising

IV) Renormalizability ? — unclear ? SUSY !

V) Expand in $\theta$ ? Use SW — not finished (QED)$_\theta$ not ren. !

VI) Recent attempts ..... ~~Last attempt~~?

Bichl
Schweda + Grimstrup
Popp
Wulkenhaar

II)

$\mathbb{R}^4_\theta$

formulation simple

Algebra: $u_p = e^{ip_\mu \hat{x}^\mu}$ $\quad [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$

smear $u_p$

$$u_p u_q = e^{i\tilde{p}q} u_{p+q}$$

Involution: $u_p^+ = u_{-p}$

Lie algebra: $[u_p, u_q] = 2i \sin(\tilde{p}q) u_{q+p}$

Diff. calculus:

derivations: $\partial_\mu u_p = i p_\mu u_p$

Leibniz: $\partial_\mu(u_p u_q) = (\partial_\mu u_p) u_q + u_p(\partial_\mu u_q)$

Integration: $\int u_p = \delta(p)$

trace: $\int u_p u_q = \int u_q u_p = \delta(p+q)$

Stokes law: $\int \partial_\mu u_p = 0$

---

Def.: Weyl op: $W(f) = \int dp \, \overbrace{e^{ip_\mu \hat{x}^\mu}}^{u_p} \tilde{f}(p)$

$\exists$ product $\ni$ $\quad W(f) W(g) = W(f * g)$

Moyal Weyl

Moyal-Weyl $(f * g)(x) = \int dk \int dp\, e^{-i(k+p)x}\, e^{+i\theta^{\mu\nu} k_\mu p_\nu}\, \tilde{f}(k)\, \tilde{g}(p)$

---

Scalar f. th.: $\int \Phi_1 \Phi_2 = \int dx\, \phi_1 * \phi_2$

F. rules $\xrightarrow{k}$ $\frac{1}{k^2+m^2}$ Filk 96

$\int \Phi_1 \dots \Phi_N = \int \prod dk_i\, \delta(\sum k_i)\, e^{i \sum_{\alpha<\beta} \theta^{\mu\nu} k_\mu^\alpha k_\nu^\beta}\, \tilde{\phi}_1(k_1) \dots \tilde{\phi}_N(k_N)$

$\Rightarrow$ planar & nonplanar diagrams

get phase factors

---

YM on $\mathbb{R}^4_\theta$

$A_\mu = \int A_\mu^p u_p$, $c = \int c^p u_p$, $\bar{c} = \int \bar{c}^p u_p$, $B = \int B^p u_p$

antic. antic.

action

$$S = \int -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \partial^\mu \bar{c} D_\mu c + \frac{\alpha}{2} B^2 + B \partial_\mu A^\mu$$

BRST invariant

$sA_\mu = \partial_\mu c + g[A_\mu, c]$, $sc = -gcc$, $s\bar{c} = B$

$sB = 0$

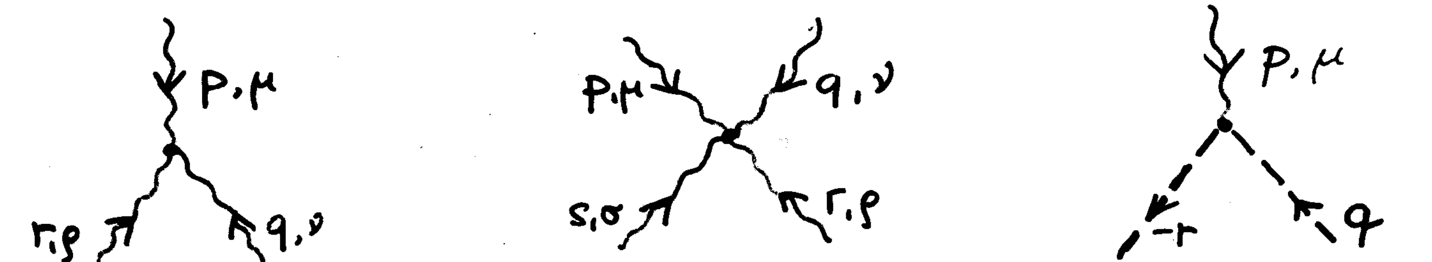
quadratic part invertible

F. rules

$p,\mu \leftarrow\!\sim\!\sim\!\sim\!\rightarrow q,\nu \qquad -\frac{g_{\mu\nu}}{p^2}\,\delta(p+q) \qquad (\alpha=1)$

$\underset{p}{\leftarrow} - - - \underset{q}{\rightarrow} \qquad -\frac{1}{p^2}\,\delta(p+q)$

nonlocal int.:




$2g\left((p^\nu - r^\nu)g^{\rho\mu} + \ldots\right)\sin\tilde{p}q\,\delta(..)$ $\qquad -4g^2\{g^{\mu\rho}g^{\sigma\nu}\sin\tilde{p}q\sin\tilde{r}s + \ldots\}\delta(..)$ $\qquad 2g\,r^\mu \sin\tilde{p}q\,\delta(..)$

---

higher loops $\qquad$ Chepelev & Roiban

Represent graph by ribbon on genus g R.S.

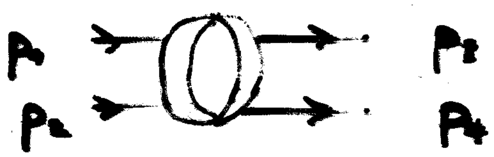
$\exists$ conv. th. $(m \neq 0)$ $\qquad$ $\exists$ degree of F.d. $\omega(\gamma)$

dangerous :

"Rings" stacked on one cycle :

? sum up $\quad 1 + \frac{a}{\tilde{p}^2} + \frac{a^2}{(\tilde{p}^2)^2} + \ldots$

"except." momenta :



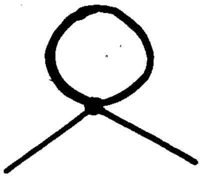

$p_1 + p_3 = 0$ $\qquad$ no regulating

$p_2 + p_4 = 0$ $\qquad$ phase

---

## III) IR/UV mixing $\phi^4$

planar



$$\int \frac{d^4k}{k^2+m^2}$$

standard BPHZ analytic reg. ...

nonplanar finite for $q \neq 0$ !!!


$$\int d^4k \, \frac{e^{2ik\tilde{q}}}{k^2+m^2} =$$

Minwalla van Raamsdonk Seiberg

$$= \int d^4k \int_0^\infty d\alpha \, \underbrace{e^{-\alpha(k^2+m^2)} \, e^{2ik\tilde{q}}}_{e^{-\alpha m^2} \, e^{-\alpha(k-\frac{\tilde{q}}{\alpha})^2} \, e^{-\frac{\tilde{q}^2}{\alpha}}} \, e^{-\frac{1}{\Lambda^2 \alpha}}$$

$$\propto \int_0^\infty \frac{d\alpha}{\alpha} \, e^{-\alpha m^2} \, e^{-\frac{1}{\alpha}\underbrace{\left(\tilde{q}^2+\frac{1}{\Lambda^2}\right)}_{\frac{1}{\Lambda_{eff}^2}}}$$

$\tilde{q}^\mu = \Theta^{\mu\nu} q_\nu$

$$q \neq 0, \Lambda \to \infty; \quad = \frac{1}{\tilde{q}^2} \underbrace{|\tilde{q}| \, K_1(2m^2|\tilde{q}|)}_{\to 1 \text{ for } q \to 0}$$

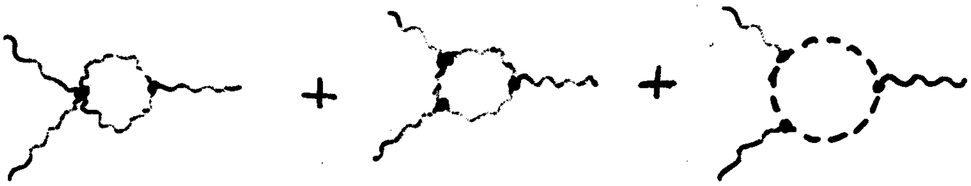
$\exists$ infrared sing. in higher loops?

$$\simeq \Lambda_{eff}^2 - m^2 \ln \frac{\Lambda_{eff}^2}{m^2} + O(1)$$

## IV)

Renormalizable ? 1 loop graphs

 $g^2 \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2}$ + reg.

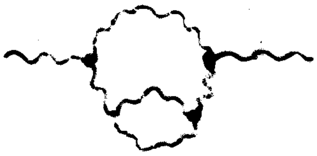
 $g^3 \cos(\tilde{p}q)\left(\frac{\tilde{p}_\mu \tilde{p}_\nu \tilde{p}_\rho}{(\tilde{p}^2)^2} + \dots\right)$ + reg.

counterterms can be absorbed by mult. ren.

on $\mathbb{R}^4_\theta$ & $T^4_\theta$

Ward/Slavnov id. ok

but $\exists$ bad IR sing., nonlocal, indep. of gauge, (and mass)

ex :  $\sim \frac{1}{k^6}$

standard BPHZ
not possible
analogous to scalar models

Logarithmic divergences would be ok

---

Arefeva, ...

Ways out : $\phi^+ * \phi * \phi^+ * \phi$

no "rings" : only even # of lines attached to boundary

"except." mom. $\rightarrow$ $\theta$-dep. mass ren.

add fermions: Wess-Zumino model

Superfields with nc coord.:

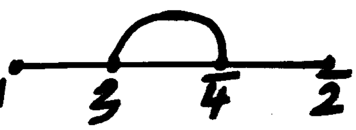
$$\phi(x,\theta) = A + \theta^\alpha \psi_\alpha + \theta^\alpha \theta_\alpha F$$

$\Phi = \int dk\, e^{ik_\mu \hat{x}^\mu}\, \tilde{\phi}(k,\theta)$ , define $*$, ...

$\rightarrow$ WZ - F. rules

One loop self energy correction

$$\int dk \, \frac{e^{ik\tilde{p}}}{(k^2-m^2+i\varepsilon)((k-p)^2-m^2+i\varepsilon)}$$




only log. IR divergences, no "ring": $\omega(\gamma) \leq 0$

even nested log. sing. are integrable

$\rightarrow$ div. come only from planar graphs

subtract by BPHZ procedure

no "except." mom.

since $\omega(\gamma) < 0$ for these graphs

V)

## YM via SW map

expand $e^{i\Theta^{\mu\nu}k_\mu p_\nu}$ to finite order

→ local f.t. non-renormaliz.

Seiberg Witten map:

between $U(1)_\Theta$ symmetry fields with hat $\hat{\lambda}, \hat{A}, \hat{F}$

and

$U(1)$ symmetry fields without hat $\lambda, A, F$

require:

$$\delta_{\hat{\lambda}[\lambda, A]} \hat{A}[A] = \hat{A}[\delta_\lambda A]$$

expand phase, deform also fields

Jurčo, Schraml, Schupp, Wess /0006246
+Madore /0001203
Way to treat nonab. negt.

Solution:

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\Theta^{\alpha\beta}A_\alpha(\partial_\beta A_\mu + F_{\beta\mu}) + O(\Theta^2)$$

$$\hat{\lambda} = \lambda - \frac{1}{2}\Theta^{\alpha\beta}A_\alpha\partial_\beta\lambda + O(\Theta^2)$$

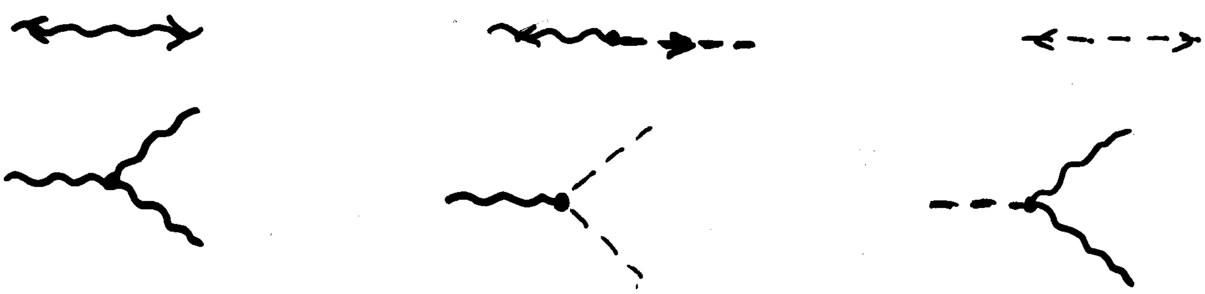
expand

$$S = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu} + \int \frac{1}{8}\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}F^{\mu\nu} + O(\theta^2)$$

vertices with many legs

Quantize : 2 ways of gauge fixing

F. rules lin. , nonlin.



One loop corr.:

$\forall$

gauge inv. cancel

not renormal. , initial action incomplete

Field redefinition

At order $n$ in $\theta$, $\exists$ nonuniqueness of solution to SW equ:

$$A_\mu^{(n)\prime} = A_\mu^{(n)} + A_\mu^{(n)} \quad \text{with}$$

$$\delta_\lambda A_\mu^{(n)} = i[\lambda, A_\mu^{(n)}]$$

effect on action

Bichl, Grimstrup, Popp, Schweda, Wulkenhaar /0102044 /0102103

$\mathcal{U}(1)_\theta$ gauge field $\rightarrow$ nc YM action

$$\Sigma_{cl} = -\frac{1}{4}\int d^4x\, \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu}$$

$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu \overset{*}{,} \hat{A}_\nu]$ inv. nc gauge transf.

$\hat{\delta}_{\hat{\lambda}} \hat{A}_\mu = \partial_\mu \hat{\lambda} - i[\hat{A}_\mu \overset{*}{,} \hat{\lambda}] \equiv \hat{D}_\mu \hat{\lambda}$

use SW $\rightarrow$

$$\Sigma_{cl} = -\frac{1}{4}\int d^4x \{F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta} F_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} + 2\theta^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} F^{\mu\nu}\} + \mathcal{O}(\theta^2)$$

inv. ab. gauge transf.

$\rightarrow$ vertices with many legs $\quad \delta_\lambda A_\mu = \partial_\mu \lambda$

---

Quantization : Two ways of

Gauge fixing :

"linear" gauge : write $sA_\mu = \partial_\mu c$ , $s\overset{\text{ghost}}{c} = 0$

$s\bar{c} = B$ , $sB = 0$

antighost N-L multipl.

add: $\Sigma^{(a)}_{gf} = \int d^4x \{ \underbrace{s(\bar{c}\partial_\mu A^\mu)}_{B\partial_\mu A^\mu - \bar{c}\partial_\mu\partial^\mu c} + \frac{\alpha}{2}B^2 \}$

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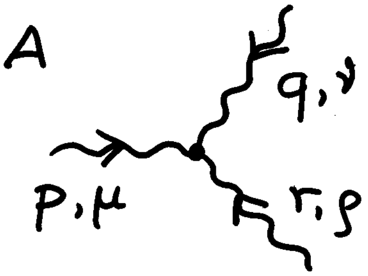
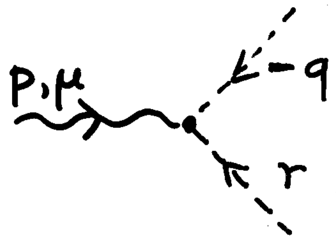
AAA



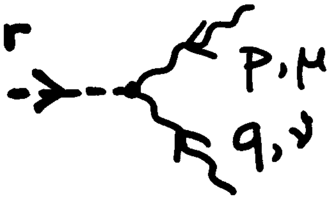

$-i\,\Theta_{\alpha\beta}\,\{ g^{\alpha\mu} g^{\beta\nu} (p,r)\, q^{\rho} - \ldots\ldots \}$

30 terms

$A\bar{c}c$




$-i\Theta_{\alpha\beta}\,\{ \tfrac{1}{2} q^2 r^{\beta} g^{\mu\alpha} + p^{\alpha} r^{\beta} q^{\mu} \}$




$\Theta_{\alpha\beta}\,\{ -\tfrac{1}{2} g^{\alpha\mu} g^{\beta\nu} (pr) + \ldots\ldots \}$

AAB

---

One loop photon self-energy    analytic reg.

$\frac{1}{\epsilon}\,\Theta^2 (p^2)^2 (g^{\mu\nu} p^2 - p^{\mu} p^{\nu}) + \ldots$

$\alpha$ indep., gauge inv., ~~[illegible]~~

$\frac{-1}{\epsilon}(p^2)^2 \tilde{p}^2 g^{\mu\nu} + \ldots$

cancel ?

$\frac{1}{\epsilon}(p^2)^2 \tilde{p}^2 g^{\mu\nu} + \ldots$

def. Maxwell action not renorm.

initial action incomplete?

"nonlinear gauge" $\qquad \hat{s}\hat{A}_\mu = \hat{D}_\mu \hat{c} \qquad \hat{s}\hat{c} = i\hat{c}\star\hat{c}$

$$\hat{s}\hat{\bar{c}} = \hat{B} \qquad \hat{s}\hat{B} = 0$$

add $\Sigma_{gf}^{(2)} = \int d^4x \left\{ \hat{s}\left(\hat{\bar{c}} \star \partial^\mu \hat{A}_\mu\right) + \frac{\alpha}{2}\hat{B}\star\hat{B} \right\}$

$$\hat{B}\star\partial^\mu\hat{A}_\mu - \hat{\bar{c}}\star\partial^\mu D_\mu \hat{c}$$

apply SW to ghost & multipl.

$$\hat{c}(c) = c - \frac{1}{2}\theta^{\mu\nu}A_\mu\partial_\nu c + O(\theta^2)$$

$$\hat{\bar{c}} = \bar{c} \qquad \hat{B} = B$$

insert $\Sigma_{gf}^{(2)} = \int d^4x \left\{ B\partial^\mu A_\mu - \bar{c}\,\partial_\mu\partial^\mu c + \frac{\alpha}{2}B^2 \right.$

$$\left. -\theta^{\alpha\beta}\left[\partial^\mu\bar{c}\,\partial_\alpha c\,\partial_\beta A_\mu - \frac{1}{2}\Box\bar{c}A_\alpha\partial_\beta c - \frac{1}{2}\partial_\mu B A_\alpha(\partial_\beta A_\mu + F_{\beta\mu})\right]\right\}$$

invariant under BRST

Both actions invariant under ab. BRST

Slavnov Taylor id: $\quad S(\Sigma^{(i)}) \equiv S(\Sigma_{cl} + \Sigma_{gf}^{(i)}) = 0$

with $\quad S(\mathcal{F}) = \int d^4x \left\{ \partial_\mu c \frac{\delta\mathcal{F}}{\delta A_\mu} + B\frac{\delta\mathcal{F}}{\delta\bar{c}} \right\}$

$\rightarrow$ F - rules

$p,\mu$  $q,\nu$ $\qquad \frac{1}{p^2+i\epsilon}\left\{ g_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2+i\epsilon} \right\} = G^{AA}$

$p,\mu$  $q$ $\qquad \frac{-ip_\mu}{p^2+i\epsilon} = G^{AB} \qquad$ $p$  $q$ $\qquad \frac{-1}{p^2+i\epsilon} = G^{\bar{c}c}$

Field redef.: At order $n$ in $\Theta$, $\exists$ nonuniqu. sol. to SW

$A_\mu^{(n)\prime} = A_\mu^{(n)} + \mathsf{A}_\mu^{(n)}$ with $\delta_\lambda \mathsf{A}_\mu^{(n)} = i[\lambda, \mathsf{A}_\mu^{(n)}]$ equ. ....

.... $S^{(n)\prime} = S^{(n)} + \int dx \, \mathrm{tr}\, \mathsf{A}_\mu D_\nu F^{\nu\mu}$

....

Renormalization of photon self-energy to all orders in $\Theta$ and $\hbar$ given by field redefinition

higher n-pt. Green's fcts. ? ? ?

calculational difficulty !

discover new symmetry ?

translations, rotations, dilations

observer & particle Lorentz transf.

... SW map

Wulkenhaar : $(QED)_{\Theta\text{-expanded}}$ is <u>not</u> renorm. ! $O(\Theta)$

compute all div. one-loop Green's fcts.

$\exists$ div. in electron four pt. fct. which cannot be removed by field redef.

---

$\to$ add $\int dx \, i\Theta^{\alpha\beta} \hat{\bar{\psi}} * \gamma^{\mu\nu}_{\alpha} \hat{D}_\beta \hat{F}_{\mu\nu} * \hat{\psi}$

+ Grimstrup

# VI Regul. through higher derivatives

$$\Delta_F(p) = \frac{1}{(p^2+m^2)(\lambda\Theta p^2+1)(\mu\Theta p^2+1)} = \sum_{i=1,2,3} \frac{A_i}{p^2+m_i^2} \qquad m_1^2 = m^2, \quad m_2^2 = \frac{1}{\lambda\Theta}, \quad m_3^2 = \frac{1}{\mu\Theta}$$

$\lambda, \mu \in \mathbb{R}$

$$\sum_i A_i = 0\,, \quad \sum_i A_i m_i^2 = 0\,, \quad \sum_i \frac{A_i}{m_i^2} = \prod_j \frac{1}{m_j^2}$$

treat $(\lambda,\mu)$ as regulator with normalization conditions:

$$\Gamma^{(2)}\Big|_{\substack{p^2=-m^2 \\ \lambda=\mu=0}} = 0\,, \qquad \frac{\partial \Gamma^{(2)}}{\partial p^2}\Big|_{\substack{p^2=-m^2 \\ \lambda=\mu=0}} = 1$$

or $\Gamma^{(2)}(p^2=p_0^2) = m^2+p_0^2$, $\frac{\partial \Gamma^{(2)}}{\partial p^2}\Big|_{p^2=p_0^2} = 1$, $\Gamma^{(4)}(\dots) = g^2$

---

for $(\lambda,\mu) \neq (0,0)$ $\rightarrow$ finite model by power counting

Claim: For $\lambda,\mu \rightarrow 0$ one loop correction terms which become singular, can be absorbed by mass renorm.:

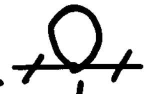
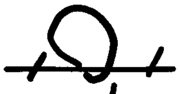
dressed prop.:

 $= \dfrac{1}{p^2+m^2-\Sigma(p)}\,, \qquad m^2 - \Sigma(p)\Big|_{p^2=-m^2} = m^2_{phys}$

$-\Sigma(p) =$  $+$  $\qquad \delta m^2 = m^2_{phys} - m^2$

pl      npl

introduce UV cutoff

$\delta m^2 = \Big[$  $_{pl}$ +  $_{npl}$ $\Big] \Bigg|_{\substack{p^2=-m^2 \\ \lambda,\mu \to 0 \\ \Lambda^2 \to \infty}}$

Claim: Can absorb all q-indep. terms in $\delta m^2$

$$\ldots \sum_i A_i \int \frac{dp\, e^{ip\tilde{q}}}{p^2+m_i^2} = \sum_i A_i \int dp\, e^{ip\tilde{q}} \int_0^\infty d\alpha\, e^{-\alpha(p^2+m_i^2)}\, e^{-\frac{1}{\Lambda^2\alpha}}$$

$$= \sum_i A_i \Big\{ \Lambda_{eff}^2 - m_i^2 \ln \frac{\Lambda_{eff}^2}{m_i^2} + O(1) \Big\}$$

$$\delta m^2 = \sum_i A_i m_i^2 \ln m_i^2$$

singular for $\lambda,\mu \to 0$
for $\Theta \to 0$

---

four point fct. more complicated ?
dominant contribution cancels too?

iterate ?

do renormalization "group" map , ....

unitarity ?

principles ?

spectral action principle ?

Summary: NC field theory

Formulation easy

Regularization possible

Renormalization (exceptional ok)

$(YM)_{nc}$ ? open

many questions:

non-constant $\Theta$ ? (which are Poisson bivect.)

causality ?

unitarity ?

results depending on vanishing electric comp. of $\Theta$ questionable ?

Renormalization puzzle ?

different normal ordering

procedures ?

principles ?