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Integrable low-dimensional theories describing high-dim. branes, black holes, cosmologies etc.

Some references: A.T.F. MPLA 11 (96) 1691; IJMPA 12 (97) 13; Fiz. Elem. Chast. i At. Yadra 32 (2001) 78 (in Russian); Yadernaya Fiz. 65 (02) 1 (Engl.)

M. Cavaglià, V. de Alfaro, A.T.F. IJMP D4 (95) 661; D5 (96) 227; Phys. Lett. B424 (98) 265

ATF + V. Ivanov: Phys. ~~Atom~~ Nucl. 61 (98) 1639

ATF + D. Maison: to be published soon

ATF + VdA: to be published

Reviews: J. Maldacena hep-th/9607235; K. Stelle: hep-th/9803116; T. Mohaupt: hep-th/0004098 — B.H. in SUGRA and superstr. theories

Toda: V.D. Ivashchuk + V.N. Melnikov: Class. Q. Gr. 18 (01) R87; P. Fré: hep-th/9909061

0. Modern development of B.H. and C.M.

- Problems with B.H. and C.M.

Statistical expalanation of B.H. thermodyn. Inform paradox. Holography. (Hawking, 't Hooft, ..)
Quantum stage of the Univ. expansion. Detailed inflation picture – explanation? $\left\{ \begin{array}{l} \text{string} \\ \text{insp. e.a} \end{array} \right.$

- 2D (1+1) models of B.H.
  CGHS, JT models (Witten, Verlinde, etc.)
- Quantizing B.H. and C.M. (simple-minded approaches) $\left\{ \begin{array}{l} \text{VdM,} \\ \text{Cavaglia} \\ \text{ATF} \end{array} \right.$
- Problem of anomalies in 1+1 dim. models (Jackiw e.a
- More complex integrable models (Korotkin, Nicolai).
- Integrable C.M. from SUGRA (Toda systems – Mel'nikov, Ivaschuk e.a P. Fre e.a)
- Non-integrable (?) C.M. from strings and branes

0'

- **Why study low-dim. D-G. (0+1, 1+1, 2+1) models?**

1. for descr. h.-d. objects – B.H., cosmologies, their evolution and interpret.
2. to have 0-th approxim. for 'non-perturbative' comput. in gravity and superstr. supercosm. (what is non-perturbative [illegible]) cosm.
3. to develop an intuitive picture of what may be Q.G. (e.g.: what is an 'oscillator' in Q.G. ?)

NB: There are return paths from L.D. to H.D. but these 'liftings' or 'oxidations' should not necessarily lead to exactly the same h.d. theory (H.N. ?)

1.

A very short history of grav. and cosmology from todays p.o.v.

• Heroic Years:

1915 Einstein, G.R.: $G_{\mu\nu} = \varkappa^2 T_{\mu\nu}$

later: Hilbert: $\mathcal{L} = \int \sqrt{-g}\, R$

• 1917 Λ - const

1916 H. Reissner; Schwarzschild

1918 G. Nordstrøm ('13 5-dim gr. + e.m)

Serini - math.

'19 Kaluza → '26 {G.A. Mandel, V.A. Fock, O. Klein}

KMFK (not KK!)

20' → Friedmann (later Lemaitre e.a.)

FRW cosmology de Sitter

'29 • Hubble! (experimental support)

• Spinors in curved spaces and in n dim

'30 → B.H. (Oppenheimer; Chandrasekhar; Landau...)

(with matter)

Necessity of Q.G. (~~Rosenfeld~~; Bohr(?))

Bad properties of Q.G. (Heisenberg)

'30 - '50 not much interest in gravity and cosmology

'37 Dirac: variable constants

3

2\. '50 → • Nonrenormaliz. of GR (perturbat)
• SUGRA (also nonrenormalizable)

• KMFK (KK) and G/H nonlinear σ-models
• Axisymm. stationary solutions
• Integrability and nonintegrability in cosmology and B.H.

• High dim. objects $\left\} \xrightarrow[\text{symm.}]{\text{compactif.}} \right\{$ Low. dim. objects
• B.H. thermod. .....

Relevance of 2D (1+1) and 1D (0+1) models of gravity to B.H. and cosmology (not the Euclidean models!)

Strings, branes, M-theory
. . . .

• Experiments!?!: • tests of GR
• existence of gr. waves (binary pulsars)
• precision checks of the B.B. picture
• B.H. observed (somewhat circumst. but strong evidence
• finite $\Lambda > 0$ (de Sitter universe)

• Conceptual problems!?! inflation; statist. expl. of B.H. therm.; inform. parad. holography ....

3. Very recent development

- String cosmology and brane cosmological models
- Quantum B.H. and Quantum Cosmological models (integrable and not integrable). "Integrability in gravity and cosmology"

* In 0+1 dim. generic integrable models are related to the Liouville and Toda theories (exceptions are indirectly related to Liouville)
* In 1+1 dim. explicitly integrable models are related to Liouville.
* Nonintegrability is interesting for cosmology and B.H. evolution but poorly understood

Main subject of these lect.

explicitly integrable models in 0+1 and 1+1 dim and their applic to Cosm., B.H., ...

• Lowest-dimensional system: the Universe!

• Homogeneous, isotropic: F-R-W cosmology

$$ds^2 = N^2(t)\,dt^2 - a^2(t)\,d_3\Omega$$ (VdA, Cavaglia, A.T.F.)

$$S = \int d^4x\,\sqrt{-g}\,(R + 2\Lambda) + S_{matter}$$ (YM, scalar, etc.)

$\Rightarrow$ (0+1) dim. system with 1 constraint

$N(t)$ – Lagrange m. $a(t)$ – grav. variable

• More general: $ds^2 = +N^2(t)\,dt^2 - h_{ij}(\bar{x},t)\,dx^i dx^j$

• homogeneous (but not isotropic)

$\Rightarrow$ h_{ij} described by a finite # of functions $q_\alpha(t)$, $\alpha = 1, \dots, n$

e.g.: spatial topology $S^1 \times S^2$ $\Rightarrow$

$$h_{ij}\,dx^i dx^j = \underbrace{a^2(t)\,dr^2}_{S^1} + \underbrace{b^2(t)\,d_2\Omega}_{S^2}$$

• SUSY extensions $\Rightarrow$ supersymmetric discrete strings, etc.

• Brane cosmology $\Rightarrow$ higher dim.

e.g. $S = \int d^5x\,\sqrt{-g}\,(R^{(5)} + 2\Lambda) - \mu_w \underbrace{\int d^4A}_{\text{invar.}}$ wall

Spherical symm. (Koyama, Soda '00 and other papers)

(1+1) DG coupled to a 'particle' (describing the domain wall)

5.

- {B.H. is a low-dim. object in high-dim. theories}
- Schwarzschild (1918) is 0+1 dimensional (R-N generalization also 0+1 dim)
- Stationary axisymm. solution of E.H. $\Rightarrow$ 2d models
- SUGRA and KMFK (KK) reductions $\Rightarrow$ G/H nonlinear σ-models + E-M (special: Liouville 2d eqs.)

E.g. $$S = \int d^dx \sqrt{-g}\left[R - \frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi - \frac{1}{2n!}e^{a\varphi}F^2_{[n]}\right]$$

$$F_{[n]} = dA_{[n-1]}$$

Spherical reduction:

$$ds^2 = g_{ij}dx^i dx^j + r^2 d\Omega^2_{(d-2)}$$

$i,j = 0,1$; $x^0 = t$, $x' = r = \left(\sum_1^{d-1} x_n^2\right)^{1/2}$

$g_{ij}(t,r)$, $\varphi(t,r)$ + corresp. Ansatz for $A_{[n-1]}$

e.g. $A_{\mu_1\ldots\mu_{n-1}} = \varepsilon_{\mu_1\ldots\mu_{n-1}} e^{A(t,r)}$

(static solut. are effectively 0+1 dim.) 7

6. General DG - matter in 1+1 dim

• $\mathcal{L}_{eff} = \sqrt{-g}\left\{ R(g)\varphi + Y(\varphi,\psi) + \sum_{k=1}^{K} Z^{(k)}(\varphi,\psi)\, g^{ij}\psi_i^{(k)}\psi_j^{(k)} \right\}$

$\psi_i^{(k)} \equiv \partial_i \psi^{(k)}(x^0, x^1)$ φ - dilaton

• Dilaton kin. en. excluded by Weyl

• Abelian gauge field included in $Y(\varphi,\psi)$

$\left(X(\varphi,\psi) F_{ij}F^{ij} \to \frac{2Q^2}{X(\varphi,\psi)} \quad (F_{ij} \equiv \partial_i A_j - \partial_j A_i) \right)$

• 0+1 dim. reduction ('static' or 'cosmolog.')

$$\boxed{\mathcal{L}_{st} = -\frac{1}{l(\tau)}\left[\dot\varphi \frac{\dot h}{h} + Z\dot\psi^2\right] + l(\tau)\, h \cdot Y(\varphi,\psi)}$$

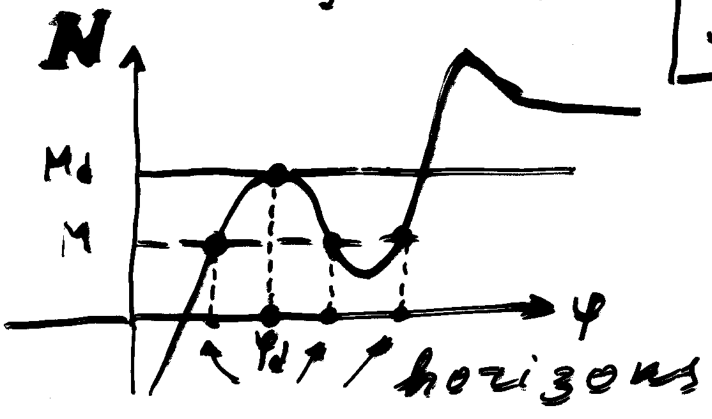
$ds^2 = -4f(u,v)\,du\,dv$ *) If $\varphi(u,v) = \varphi(\underbrace{a(u)+b(v)}_{\tau})$

$\Rightarrow f(u,v) = h(\tau)a'(u)b'(v)$, $\Rightarrow$ $ds^2 = \underbrace{-4h(\tau)\,da\,db}$ $= h(\tau)(dr^2 - d\tau^2)$

*) In (u,v) coord. $R(g) = \frac{1}{f}(\ln|f|)_{uv}$

If $Z \equiv 0$, $Y \equiv V(\varphi)$, we have the general solution $\boxed{h = M - N(\varphi)}$

$N(\varphi) \equiv \int d\varphi\, V(\varphi)$ $\int \frac{d\varphi}{M - N(\varphi)} = \tau - \tau_0$ $(\Rightarrow \varphi(\tau))$



φ_d - "degenerate" horizon

$N'(\varphi_d) \equiv V(\varphi_d) \equiv 0$

* From high to low dimension:

$$\mathcal{L}^{(10)} = \mathcal{L}^{(10)}_{NS-NS} + \mathcal{L}^{(10)}_{RR} \qquad \underline{G = \det(G_{\mu\nu})}$$

$$\mathcal{L}^{(10)}_{NS-NS} = \sqrt{-G}\, e^{-2\phi_s}\left[R^{(10)} + 4(\nabla\phi_s)^2 - \frac{1}{12}H_3^2\right]$$

$$H_3 = dB_2 \; ; \quad F_2^2 = (\nabla f)^2 \qquad \text{Note signs!}$$

$$\mathcal{L}^{(10)}_{R-R} = \sqrt{-G}\left\{\left[-\frac{1}{4}F_2^2 - \frac{1}{48}F_4^2\right]_{IIA} + \right.$$
$$\left. + \left[-\frac{1}{2}F_1^2 - \frac{1}{12}F_3^2 - \frac{1}{240}F_5^2\right]\right\}_{IIB}$$

(Ex) Different dim reductions (K-M-K-F [KK], compactif on tori, ...)

$$\mathcal{L}^{(d)} = \sqrt{-G}\, e^{-2\phi_d}\left[R^{(d)} + 4(\nabla\phi_d)^2 - \frac{1}{2}(\nabla\psi)^2 - \right.$$
$$\left. - X_0 - X_1(\nabla\sigma)^2 - X_2 F_2^2 - \ldots\right], \quad X_i = X_i(\phi_d, \psi)$$

(several ψ, σ) independent of σ

(App.) Weyl transformation: $G_{\mu\nu} = +\Omega^{-2}\bar{G}_{\mu\nu}$

$$\mathcal{L}^{(d)} = \sqrt{-\bar{G}}\left[\Omega^{2-d}\cdot e^{-2\phi_d}\right]\cdot\left[\bar{R}^{(d)} - 4\nu(\nabla\phi_d)^2 - \right.$$
$$- \frac{1}{2}(\nabla\psi)^2 - X_1(\nabla\sigma)^2 e^{2\phi_d} - X_2 e^{-4\omega\phi_d + 2\phi_d} F_2^2$$
$$\left. - X_0 e^{4\omega\phi_d + 2\phi_d}\right]$$

Choose: $\underline{\left[\Omega^{2-d} e^{-2\phi_d}\right] = 1}$

↳ 'Einstein frame'

Ex. 07 Rep.

★ From 11d → (1+1)d (SUGRA → DGM)

Bosonic sector:

• $$S^{(11)} = \int dx^{11} \left[\sqrt{-g} \left(R - \frac{1}{48} F_{[4]}^2 \right) + \frac{1}{6} F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \right]$$

$F_{[4]} = d A_{[3]}$ $\quad S^1 \to$ compactif $\quad$ II A $d=10$ $\updownarrow$ T duality II B $d=10$

• $$\Rightarrow S^{(10)} = \int dx^{10} \sqrt{-g} \left[R(g) + 4(\nabla \varphi^{(10)})^2 - \frac{1}{12} H_{[3]}^2 \right] e^{-2\varphi^{(10)}}$$

φ – dilaton $\qquad$ NS-NS sector of II A th. (the same for II B)

Furter reductions: e.g. K-K, compact. on tori, ...

Brane*) + K.K • $ds^2 = g^{(6)}_{\mu\nu} dx^\mu dx^\nu + e^{2\psi} dx^m dx_m$

$$g^{(6)}_{\mu\nu} = \begin{pmatrix} g^{(5)}_{\alpha\beta} + e^{\psi_1} A_\alpha A_\beta & e^{\psi_1} A_\alpha \\ e^{\psi_1} A_\beta & e^{\psi_1} \end{pmatrix} \qquad m, n = 6, 7, 8, 9$$

$\psi, \psi_1, F_{\mu\nu} = dA, \tilde{F}_{\mu\nu}$ $\overset{"}{H}_{\mu\nu 5}$

⇒ 5-dim gravity + dilaton + matter

• $$\mathcal{L} = \sqrt{-g^{(5)}}\, e^{2\varphi_1} \left\{ R^{(5)} + 4(\nabla\varphi)^2 - 4(\nabla\psi)^2 - \frac{1}{4}(\nabla\psi_1)^2 - \frac{1}{12} H'^2 - \frac{1}{4}\left(e^{-\psi_1} \tilde{F}^2 + e^{\psi_1} F^2 \right) \right\}$$

$\varphi_1 = \varphi^{(10)} + \psi + \frac{1}{4}\psi_1$

· Sph. symm.: • $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta + e^{-2\psi_2} d\Omega^2_{(3)}$

• $$\Rightarrow \text{DGM}: \ \mathcal{L} = \sqrt{-g} \left\{ R^{(2)} + (\alpha e^{2\psi_2} + \beta e^{6\psi_2}) + 4(\nabla\varphi)^2 - 4(\nabla\psi)^2 - \frac{1}{4}(\nabla\psi_1)^2 - 3(\nabla\psi_2)^2 - \frac{1}{4}\left(e^{-\psi_1} F^2 + e^{\psi_1} \tilde{F}^2 \right) \right\} e^{-2\varphi}$$

*) Rem. $\quad A_{[n-1]} : A_{\mu_1 \ldots \mu_{n-1}} =: \varepsilon_{\mu_1 \ldots \mu_{n-1}}\, e^{A(r,t)}$ (electric)

for spherical symm. reduction on a p-brane (similarly - magnetic field) app. **Weyl transformation.**

$$\begin{cases} \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}\,, \quad d\tilde{s}^2 = \Omega^2 ds^2 \\ \sqrt{-\tilde{g}} = \Omega^D \sqrt{-g} \end{cases}$$

$g \equiv G$!

$$\left\{ R(g) = R^{\mu}{}_{\mu}(g) \right.$$

$$R \equiv R(g) = \Omega^2 \Big[\tilde{R} + 2(D-1)\,\tilde{\Box} \ln \Omega -$$

$D = d$!

$$- (D-2)(D-1)\,\tilde{g}^{\mu\nu} \frac{\Omega_{,\mu}\Omega_{,\nu}}{\Omega^2} \Big]$$

Let

$$S = \int_M d^D x \sqrt{-g}\, e^{-\varphi} \left[R - \omega (\nabla\varphi)^2 - .. \right]$$

$$+ 2\int_{\partial M} ... \sqrt{|h|}\, \underset{\sim}{K}$$

K = extrins. curv.
(important for embeddings)

$$S = \int_M d^D x \sqrt{-\tilde{g}}\, e^{-\varphi}\, \Omega^{2-D} \Big[\tilde{R} - \omega(\tilde{\nabla}\varphi)^2 +$$

$$+ (D-2)(D-1)\,\Omega^{-2} (\tilde{\nabla}\Omega)^2 + 2(D-1)\,\Omega^{-1} \cdot (\tilde{\nabla}\Omega)(\tilde{\nabla}\varphi) \Big]$$

$$+ 2 \int_{\partial M} \sqrt{|h|}\, e^{-\varphi}\, \Omega^{2-D} \tilde{K}$$

Let:

$$\Omega = e^{\varphi} : \quad \boxed{S = \int \sqrt{|\tilde{g}|} \left[\tilde{R} - \left(\omega + \frac{D-1}{D-2}\right)(\nabla\varphi)^2 \ldots \right]}$$

$D \neq 2$

$d-2=n$, $\nu = 1/n$, $d \neq 2$:

$$\mathcal{L}_E^{(d)} = \sqrt{-G}\Big[\bar{R}^{(d)} - 4\nu(\nabla\phi_d)^2 - \frac{1}{2}(\nabla\psi)^2 - X_0 e^{2\phi_d\cdot\nu d} - X_1 e^{2\phi_d}(\nabla\sigma)^2 - X_2 e^{2\phi_d\cdot\nu(d-4)}\cdot F_2^2\Big]$$

Renormal. ϕ_d and introducing new notat.

$$\mathcal{L}_E^{(d)} = \sqrt{-G}\Big[R^{(d)} - \frac{1}{2}(\nabla\phi_d)^2 - \frac{1}{2}(\nabla\psi)^2 - X_0 e^{a_0\phi_d} - X_1 e^{a_1\phi_d}(\nabla\sigma)^2 - X_2 e^{a_2\phi_d} F_2^2\Big]$$

a typical dim. reduced Lagr.

* Next step: a <u>symmetry reduction</u>

- Spherical symmetry reduction

$$ds^2 = g_{ij}\,dx^i dx^j + e^{-4\nu\phi_2}\,d\Omega_n^2$$

$(i,j=0,1)$ all functions depend on (x^0, x^1) (t, r)

$$\Rightarrow \mathcal{L}^{(2)} = \sqrt{-g}\,\underbrace{e^{-2\phi_2}}_{\text{not removable}}\Big[R^{(2)} + n(n-1)e^{4\nu\phi_2} + \underbrace{4(1-\nu)(\nabla\phi_2)^2}_{\text{removable}} - X_2 e^{a_2\phi_d} F_2^2 - \frac{1}{2}(\nabla\phi_d)^2 - \frac{1}{2}(\nabla\psi)^2 - X_1 e^{a_1\phi_d}(\nabla\sigma)^2 - X_0 e^{a_0\phi_d}\Big]$$

removable by a Weyl transformation

$$\mathcal{L}^{(2)} = \sqrt{-g}\Big[\varphi R + n(n-1)\,\varphi^{-\nu} - X_2 e^{a_2\chi}\varphi^{2-\nu}F^2 -$$
$$- X_0 e^{a_0\chi}\varphi^{\nu} - \frac{\varphi}{2}\Big((\nabla\chi)^2 + (\nabla\psi)^2 + 2X_1 e^{a_1\chi}(\nabla\sigma)^2\Big)\Big]$$

Note signs! $(X_i \geq 0)$ $\quad n(n-1) \geq 0$

$$\chi \equiv \phi_d \qquad \varphi \equiv e^{-2\phi_2}$$

This is a typical DGM — (dilaton gravity coupled to matter) related to SUGRA and superstrings (also branes)

General DGM theory in 1+1 dim:

$$\mathcal{L} = \sqrt{-g}\Big[U(\varphi)R(g) + V(\varphi) + \underbrace{W(\varphi)(\nabla\varphi)^2}_{\text{removable}}$$
$$+ \sum_a X_a(\varphi,\psi)F_{(a)}^2 + Y(\varphi,\psi) + \sum_n Z_n(\varphi,\psi)(\nabla\psi^{(n)})^2\Big]$$

! (U,V,W)*) — theory is DG (dilaton gravity)
Explicitly integrable for arb. U,V,W

! (V,W,W,X,Y,Z)-th. generally not integrable

Using Weyl and excluding $F^{(a)}_{ij} \equiv \partial_i A_j - \partial_j A_i$

$$\Rightarrow \mathcal{L} = \sqrt{-g}\Big[U(\varphi)R(g) + Y_{eff}(\varphi,\psi) + \sum_n Z_n(\varphi,\psi)(\nabla\psi^{(n)})^2\Big]$$

$$Y_{eff} \equiv Y(\varphi,\psi) + \sum_a \frac{2Q_a^2}{X^{(a)}(\varphi,\psi)}, \quad Q_a - \text{el. charg.}$$

(magnetic charges — similarly)

*) If $X_a = X_a(\varphi)$, $(UVWX)$-theory is also integrable.

• From 1+1 to 0+1 dimension

LC metric : $\boxed{ds^2 = -4f(u,v)\,du\,dv}$

'Static' solutions and cosmological models :

$\phi(u,v) = \phi(\tau)$, $\boxed{\tau \equiv a(u) + b(v)}$

$\rightarrow \underline{f(u,v) = h(\tau)\, a'(u)\, b'(v)}$, $\dashrightarrow \psi(u,v) = \psi(\tau)$

'static' eqs. in general not integrable

• Solution of DG (no scalar matter)

Eqs

$U(\phi) \equiv \varphi$, $W \equiv 0$ (by Weyl)

$Y_{eff} = V(\varphi)$, $Z_n \equiv 0$

$$\left.\begin{array}{l} h(\tau) = -\int d\varphi\,[V(\varphi)] \\ \quad + M \end{array}\right\} ; \quad \int d\varphi\,[M - N(\varphi)]^{-1} = \tau - \tau_0$$

$$N(\varphi) \equiv \int d\varphi\, V(\varphi)$$

(integrals M, Q_a, τ_0 ...)

[Actually, this is the general solution of DG in dim $1+1$

We may (in this case) reduce 1+1 to 0+1, find $g_{ij}(\tau)$, $\phi(\tau)$, $A(\tau)$ and ~~then~~ take $\tau = a(u) + b(v)$ with arbitrary $a(u)$, $b(v)$ ■ $\boxed{h = M - N(\varphi) = 0}$ horizon

Eqs

Eqs (Rep) (1a)

* Equations of motion in (u,v)

Constraints: $\boxed{f\left(\frac{\varphi_i}{f}\right)_i = Z\psi_i^2 \; ; i=u,v}$
(lin. comb. of $\mathcal{E}$ and $\mathcal{P}$)

$$\varphi_{uv} + f(Y_{eff}) = 0 \quad (A)$$

$$(*)\quad (Z\psi_u)_v + (Z\psi_v)_u + fY_{eff,\psi} = Z_\psi \psi_u \psi_v$$

$$(\ast\ast)\quad (\log|f|)_{uv} + fY_{eff,\varphi} = Z_\varphi \psi_u \psi_v$$

$(*)$ and $(\ast\ast)$ are 'dependent'.
(we may use $(*)$ and then $\Rightarrow (\ast\ast)$, and v.v.)

* If $Z=0$: $\left(\frac{\varphi_u}{f}\right)_u = 0 \Rightarrow \frac{\varphi_v}{f} = \frac{1}{b'(v)}$

$$\left(\frac{\varphi_v}{f}\right)_v = 0 \Rightarrow \frac{\varphi_v}{f} = \frac{1}{a'(u)} \Rightarrow \varphi = \varphi(a+b)$$

$$f = h\, a'(u)\, b'(v)\,, \quad h \equiv \varphi'(\underbrace{a+b}_{=\tau})$$

* If $Y_{eff} \equiv V(\varphi) \equiv N'(\varphi) \xRightarrow{(A)}$

$$\varphi''(\tau) + \varphi'(\tau) N'(\varphi) = 0$$
$$[\varphi' + N(\varphi)]' = 0$$

B.H. mass

$$\varphi' \equiv h = M - N(\varphi)$$

Sol. 2

- No static horizon if $\left(\frac{\varphi_i}{f}\right)_i \neq 0$

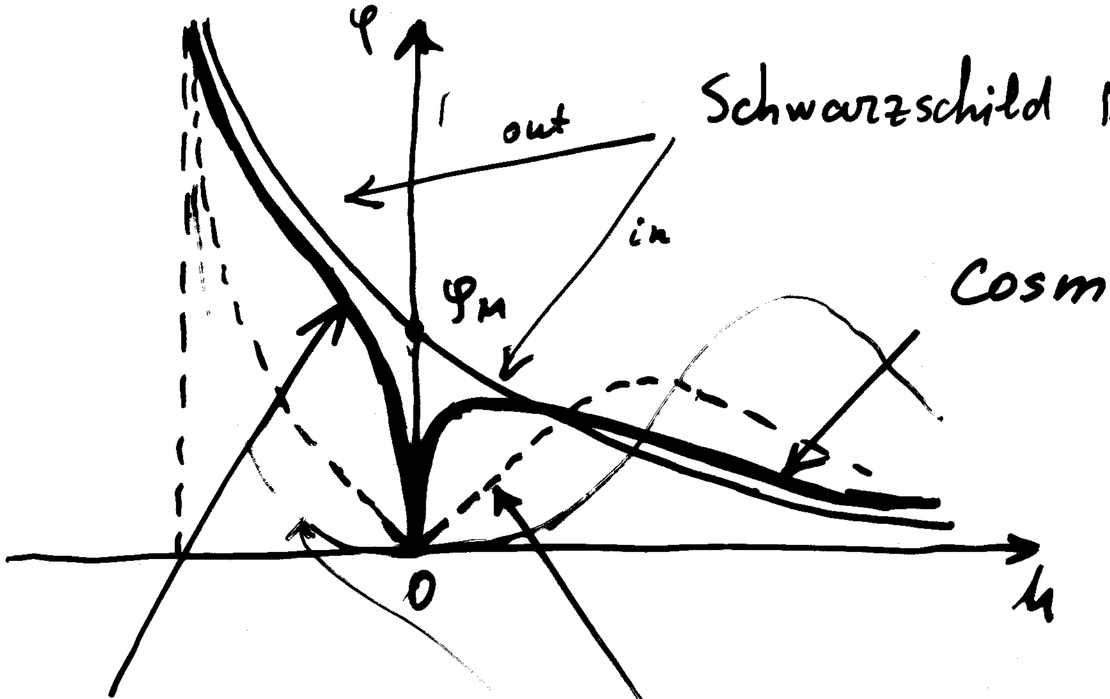
$Y \equiv V(\varphi),\ Z = Z(\varphi)$

More precisely: if $Z(\varphi) \neq 0,\ \neq \infty$

(sufficient that $Z \neq \infty$) $V(\varphi) \neq \infty$

Then there is no static horizon,

i.e. $h(\varphi) \neq 0$



Static distrib. of the scalar matter (small energy of the scalar field)

Bifurcation (between high and low scalar field en.)

'Strong' scalar field

- A typical picture of DGM solutions (in integrable case)

1

$$\mathcal{L} = \sqrt{-g}\left(\varphi R + V(\varphi,\psi) + Z(\varphi,\psi)(\nabla\psi)^2\right)$$

$$\varphi = e^{\Phi}, \quad V \equiv Y_{eff}(\varphi,\psi)$$

constant fields

1. $(V = V(\varphi))$, $\nabla\psi \equiv 0$ (or $Z \equiv 0$)

all solutions are static.

1a Horizons.

2. $V = V(\varphi)$, $Z = Z(\varphi)$ $\nabla\psi \not\equiv 0$

Static solutions have no horizons. ('NO HAIR)

- In general not integrable

(cosmological)

<u>Static</u>$^{\tau}$ sector: $\varphi = \varphi(\tau)$

$$f = h(\tau)\, a'(u)\, b'(v), \quad \tau = a(u) + b(v)$$

$$ds^2 = -4 f(u,v)\, du\, dv$$

$$\mathcal{L}_{st} = -\frac{1}{\ell}\left(\dot\varphi \frac{\dot h}{h} + Z\dot\psi^2\right) + \ell h V(\varphi,\psi)$$

$$\dot\varphi \equiv \frac{d\varphi}{d\tau}$$

(6 terms: [illegible] independent of [illegible])

Sol. 4. (2)

<u>an example</u> : $V(\varphi,\psi) = 2g\psi^n$

$$\mathcal{L} = -\frac{1}{\ell}\left(\dot\varphi\frac{\dot h}{h} + Z\dot\psi^2\right) + \ell h \cdot 2g\psi^n$$

$(n=1;\ n=2)$

$n=1$ is very easy to solve

general. $$\begin{cases} \psi = \psi_0 + \psi_1 T + g\beta^{-2} e^{\alpha+\beta T} \\ h = \exp(\alpha+\beta T), \quad T \equiv \int \ell(\tau)\, d\tau \end{cases}$$

a horizon exists if and o.if

$$\psi_1 = 0$$

Then $$\boxed{\varphi = \varphi_0 - 2gh\psi_0\beta^{-2} - g^2h^2(2\beta^4)^{-1}}$$

One of the 2 solutions gives a horizon

$$\begin{cases} h = -\dfrac{\varphi-\varphi_0}{g\psi_0}\left(1+\sqrt{1-\dfrac{\varphi-\varphi_0}{2\beta^2\psi_0^2}}\right)^{-1} \\ h = 0 \text{ for } \varphi = \varphi_0 \end{cases}$$

* General equations of DGM in 0+1

$$\mathcal{L}_{st} = -\frac{1}{\ell(\tau)}\left[\varphi' h' h^{-1} + Z\psi'^2\right] + \ell(\tau)\left[h\, Y(\varphi,\psi)\right]$$

Lagrange multiplier (metric coeff.)

1 field for simplicity

Lagrange or Hamilton eqs. are easy to write

[No horizon theorem: φ is the dim. of sph.
if $Y_\psi \equiv 0$. then $h(\varphi_0) \neq 0$ for $0 < \varphi_0 < \infty$ and $\psi \neq$ const.

If $Y_\psi \neq 0$ then we always have a horizon, $h(\varphi_0) = 0$

Near the horizon we have locally convergent expansions (D. Maison, ATF)

$$\begin{cases} \varphi = \varphi_0 + \varphi_1 x + \varphi_2 x^2 + \dots \\ h = h_0 x\,(1 + h_1 x + h_2 x^2 + \dots) \\ \psi = \psi_0 + \psi_1 x + \psi_2 x^2 + \dots \end{cases}$$

$$\left(x \equiv \dot{\varphi} \equiv \frac{d\varphi}{dT} \qquad (T \equiv \int \ell(\tau)\, d\tau)\right)$$

h_0, ψ_0 arbitrary; $h_1 = \frac{Y_\psi^2 Z_0}{4h_0 Y^3}$ $\quad \varphi_1 = -\frac{1}{h_0 Y}$

φ_n, ψ_n, h_n can be computed recursively ...

The horizon emerges at $T \to \infty$

$$x = e^{\frac{T - T_0}{\varphi_1}}\left(1 - 2\frac{\varphi_2}{\varphi_1} e^{(T-T_0)/\varphi_1} + \dots\right)$$

• Integrable theories in 0+1 dim

I. N-Liouville models   Consider a class of models

$$\mathcal{L} = \frac{1}{l}\left(-\dot F\dot\varphi - \sum_{n=3}^{N} z_n \dot\psi_n^2\right) + l\varepsilon \sum_{n=1}^{N} \frac{1}{2} g_n e^{q_n}$$

$$q_n = \sum_{m=1}^{n} \psi_m a_{mn} \; ; \quad \psi_1 = \frac{1}{2}(F+\varphi), \quad \psi_2 \equiv \frac{1}{2}(F-\varphi)$$

Suppose that $z_n = -1$ $\left[\begin{array}{l} a_{1n} = 1 + a_n \\ a_{2n} = 1 - a_n \end{array}\right.$

Suppose that a_{mn} satisfy the relations ($\varepsilon_1 = -1$, $\varepsilon_m = 1$, $m \geq 2$)

$$\sum_{m=1}^{N} \varepsilon_m a_{ml} a_{mn} = \frac{1}{\gamma_n} \delta_{ln}$$

Then the e.o.m. are reduced to N Liouville equations

$$\boxed{\ddot q_n + \bar g_n e^{q_n} = 0} \qquad \bar g_n = -\frac{g_n}{\gamma_n}\varepsilon$$

($\bar g_n \gtrless 0$ or $\bar g_n = 0$) $\varepsilon \equiv h/|h|$.

$$\boxed{e^{-q_n} = \frac{|\bar g_n|}{2\mu_n^2}\left(e^{\mu_n(\tau-\tau_n)} + e^{-\mu_n(\tau-\tau_n)} + 2\varepsilon_n\right)}$$

$\mu_n^2 \in \mathrm{Re}$ $\varepsilon_n \equiv \bar g_n / |\bar g_n|$

The constraint: $\boxed{\sum_{n=1}^{N} \gamma_n \mu_n^2 \equiv 0}$

Moduli: $\mu_1, \dots, \mu_{N-1}; \; \tau_1, \dots, \tau_{N-1}$

• One and only one γ_n is < 0 (γ_1)

$\sum \gamma_n \equiv 0$

• Horizons in N-Liouville.

I If $|\tau| \to \infty$, $q_n \to -\infty$ ($\mu_n \in \mathrm{Re}$)
but scalar fields may be finite for special relations between moduli μ_n.

$h = e^F$

A) $F \to -\infty$ ($h \to 0$), $\left\{ \underline{\phi \to \phi_0} \right.$ – horizon, $R \to R_0$

B) $F \to F_0$, $\underline{\phi \to \pm\infty}$ $\left\{ \text{flat space}, \; R \to 0 \right.$

II If $|\tau - \tau_n| \to 0$, $\varepsilon_n < 0$, $\forall n$

$q_n \to +\infty$ Singularity!

However, for $\tau_n \equiv \tau_0$ $\forall \psi_n$ are finite!

$F \to +\infty$, $\phi \to \phi_0$, $R \to R_c$

$$\boxed{R^{(2)} = \frac{1}{f}(\log f)_{uv} \xrightarrow{\text{'st.'}} \frac{1}{h}(\log h)''}$$

(no singularity and R is finite)

There exist nonsingular solutions!

It is not yet clear whether such models may be found among dim. reduced high-dim. theories.

| Also interesting cosmological models!

- There exist also models not reducible to N-Liouville but integrable

Consider the above example

$$\mathcal{L}^{(1)} = \frac{\varphi}{\ell}\left(-\frac{\dot\varphi}{\varphi}\dot F + \frac{1}{2}(\dot x^2+\dot\psi^2) + X_1 e^{a_1 x}\dot\sigma^2\right) + \underbrace{\frac{\ell}{\varphi}}_{=\bar\ell}\varepsilon e^F\left(n(n-1)\varphi^{1-\nu} - X_3 e^{a_3 x}\varphi^{1+\nu} - \frac{2Q^2}{X_2}e^{-a_2 x}\varphi^{-(1-\nu)}\right)$$

$\varphi \equiv e^{\phi}$ and integrate out X_1-term (σ-term)

$$\mathcal{L}^{(1)}_{eff} = \frac{1}{\ell}\left(-\dot\phi\dot F + \frac{1}{2}(\dot x^2+\dot\psi^2)\right) + \bar\ell\varepsilon \underset{\sim}{e^F}\left(n(n-1)e^{(1-\nu)\phi} - X_3 e^{a_3 x + (1+\nu)\phi} - \frac{2Q^2}{X_2}e^{-a_2 x - (1-\nu)\phi}\right) + \underset{\sim\sim\sim}{\bar\ell\left(-\frac{c_1^2}{X_1}e^{-a_1 x}\right)}$$

Note a different F-dependence

Such theories may be integrable if equiv. to N-Liouville or Toda (or to some degenerate theories)

Otherwise, they are (most probably) NOT INTEGRABLE.

Integrability is a miracle (explained by some symm)

- Much less miracles in 1+1 (but still there are some!)

If $Z_n(\varphi, \psi) = \text{const}$ and if the 0+1 reduction of the theory is N-Liouville we have explicitly integrable 1+1 dim. theories reduced to Liouville $q^{(n)}_{uv} + \bar{g}_n e^{q^{(n)}} = 0$

A more complex story!

The main problem is to solve the constraints.

<u>In Brief</u>:
$$\begin{cases} \varphi_{uv} + fY = 0 \\ (Z\psi_u)_v + (Z\psi_v)_u + fY_\psi = Z_\psi \psi_u \psi_v \end{cases}$$

$$\boxed{C_i = f\left(\frac{\varphi_i}{f}\right)_i - Z\psi_i^2}, \quad i = u, v; \quad \begin{cases} \psi_u = \frac{\partial \psi}{\partial u} \\ \psi_v = \frac{\partial \psi}{\partial v}, \ldots \end{cases}$$

Nonlinear part. diff. equations

However <u>we can (explicitly) solve them in terms of some arbitrary chiral functions</u> $\left\{ a_n(u),\ b_n(v) \right.$ $2 \le n \le N$ (for N-Liouville)

Applications of the Classical Solutions

1) 0+1 dim solutions BH and CM
The same theory may describe both objects.

2) 0+1 dim. solutions may be ***imbedded*** into 1+1 dim solutions of 1+1 dim. integrable models.

Possible: 1. to describe evolution of scalar matter configurations to B.H. and forming horizons — analytic, not numeric.

2. to describe cosmologies which are homogeneous but not isotropic.

3. ~~to~~ describe analytic relations between cosmologies and B.H. (horizons).

3) To ~~describe~~ construct perturb. theory on the integrable backgr. (both in 0+1 and 1+1 dimension)

• Quantum theory ?

• Simple for 0+1 dim. integrable

DGM• Many ways leading to quantizing 0+1 dim. sectors (D.G. is completely worked out by VDA, M.Cav. and ATF) Quantm horizons?

• For non-integrable 0+1 dim DGM some possibilities for quantizing near horizons (?)

• Quantum theory in 1+1 dim. : integrable N-Liouville models — some chances to quantize in near future (using quantum theory of the Liouville eq.; the main problem — quantum constraints and technical complexity of quantum Liouville on non-compact surfaces)

• Note: Quantum cosmology seems to be in a better shape than QBH.