p-ADIC AND ADELIC METHODS IN STRING THEORY AND QUANTUM COSMOLOGY

Kopaonik, 2002.

Branko Dragovich

- Introduction
- p-Adic numbers and adeles
- Motivations and approach
- p-Adic and adelic string theory
- p-Adic and adelic quantum cosmology
- Conclusion

• Introduction

p-adic string Volovich (1987)

Arefeva, B.D., Vladimirov, Khrennikov, ...
Freund, Witten, Frampton, ...

$$\Delta x \geq \ell_0 = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33}\,\mathrm{cm}$$

Some recent papers

Vladimirov: math-phys/0004017

Ghoshal, Sen: hep-th/0003278
Minahan: hep-th/0105312
hep-th/0102071
Moeller, Zwiebach: hep-th/0207107

Djordjevic, B.D., Nesic, Volovich:
"p-ADIC AND ADELIC MINISUPERSPACE QUANTUM COSMOLOGY", Int. J. Mod. Phys. A **17** (2002) 1413-1433

p-ADIC NUMBERS AND ADELES

$\mathbb{Q}$ = field of rational numbers

$x \in \mathbb{Q}$

REAL

$$x = \pm \sum_{n=n_0}^{-\infty} a_n 10^n$$

$a_n = 0, 1, \ldots, 9$

ordinary absolute value

$|x|_\infty$

$d_\infty(x,y) = |x-y|_\infty$

$$|x|_\infty = \begin{cases} x, & x \geqslant 0 \\ -x, & x < 0 \end{cases}$$

P-ADIC

$$x = \sum_{m=m_0}^{+\infty} b_m p^m$$

$b_m = 0, 1, \ldots, p-1$

p-adic abs. value (p-adic norm)

$|x|_p$

$d_p(x,y) = |x-y|_p$

$$x = \frac{m}{n} = p^\nu \frac{a}{b}$$

$|x|_p = p^{-\nu}$, $|0|_p = 0$

$$\boxed{|x|_\infty \prod_p |x|_p = 1}$$

$x \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$

$|x+y|_p \leq \max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$

strong triangle inequality

nonarchimedean (ultrametric) norm !

$| \ | : F \longrightarrow \mathbb{R}_{+}$

1. $|x| \geq 0$, $\quad |x| = 0 \longleftrightarrow x = 0$
2. $|xy| = |x|\,|y|$
3. $|x+y| \leq |x| + |y|$

$|x+y|_\infty \leq |x|_\infty + |y|_\infty$ archimedean norm

$$|x+y|_p \leq \max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$$

$\uparrow$
nonarchimedean (ultrametric) norm

Examples

$x = \frac{15}{2} = \frac{3 \cdot 5}{2}$

$\left|\frac{15}{2}\right|_2 = 2$, $\left|\frac{15}{2}\right|_3 = \frac{1}{3}$, $\left|\frac{15}{2}\right|_5 = \frac{1}{5}$,

$\left|\frac{15}{2}\right|_p = 1 \quad p \geq 7$.

$$\boxed{|m|_p \leq 1, \quad m \in \mathbb{Z}}$$

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \dots$$
$$= \sum_{n=0}^{+\infty} (p-1)p^n$$

All valuations on $\mathbb{Q}$

$(\mathbb{Q}, |\cdot|_\infty)$

$(\mathbb{Q}, |\cdot|_2)$

$(\mathbb{Q}, |\cdot|_3)$

$\vdots$

$(\mathbb{Q}, |\cdot|_\infty)$

$(\mathbb{Q}, |\cdot|_p)$

$p = 2, 3, 5, 7, 11, \ldots$

Ostrowski theorem: Each non-trivial valuation on $\mathbb{Q}$ is equivalent either to $|\cdot|_\infty$ or $|\cdot|_p$.

Completions of $\mathbb{Q}$

$$\begin{tikzcd} & \mathbb{R} \equiv \mathbb{Q}_\infty \\ \mathbb{Q} \arrow[ur, "|\cdot|_\infty"] \arrow[dr, "|\cdot|_p"'] & \\ & \mathbb{Q}_p \end{tikzcd}$$

$p = 2, 3, 5, \ldots$

$x \in \mathbb{R}$: $\quad x = \pm(x_n 10^n + \ldots + x_0 + x_{-1} 10^{-1} + \ldots)$

$x \in \mathbb{Q}_p$: $\quad x = x_{-k} p^{-k} + \ldots + x_0 + x_1 p^1 + x_2 p^2 + \ldots$

expansion into oposite directions

$|x|_p = p^k$

REAL NUMBER

$$x = \pm \sum_{n=k}^{-\infty} a_n 10^n$$

$$= \pm (a_k 10^k + \dots + a_0 + a_{-1} 10^{-1} + \dots)$$

P-ADIC NUMBER

$$x = \sum_{n=-k}^{+\infty} b_n p^n$$

$$= b_{-k} p^{-k} + \dots + b_0 + b_1 p + b_2 p^2 + \dots$$

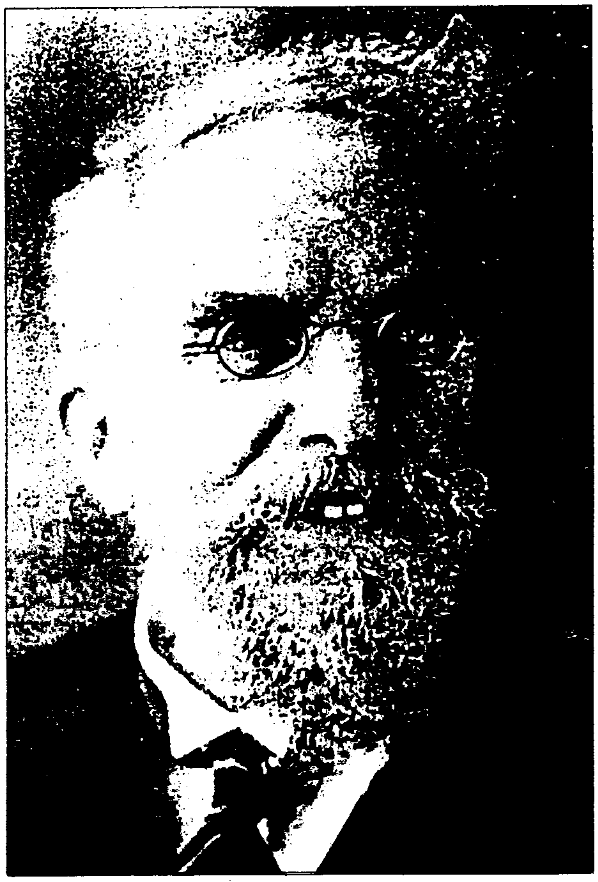
$$b_i = 0, 1, \dots, p-1$$

$\mathbb{Q}_p$ = field of p-adic numbers

$\mathbb{Z}_p$ = ring of p-adic integers

$$\mathbb{Z}_p = \{ x \in \mathbb{Q}_p : x = x_0 + x_1 p + \dots \}$$

$\mathbb{Z}$ is dense in $\mathbb{Z}_p$!



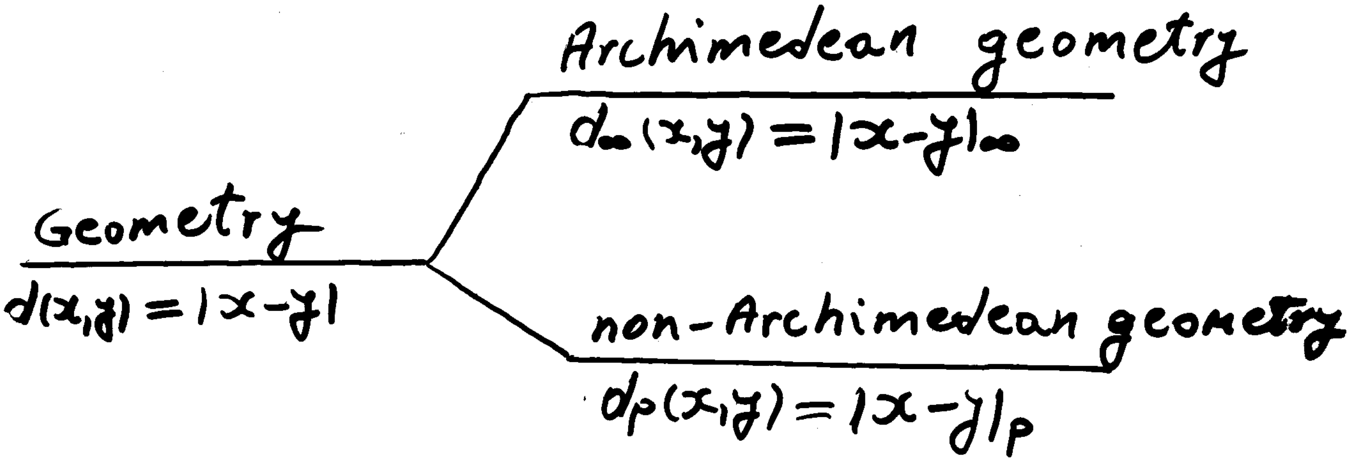
Le mathématicien allemand ***Kurt Hensel*** *(1861-1941) inventa les nombres p-adiques, au début du XXᵉ siècle. Il était un élève du célèbre théoricien des nombres Leopold Kronecker. Hensel enseigna à Berlin, puis à l'université de Marburg. (Cliché Jean-Loup Charmet)*

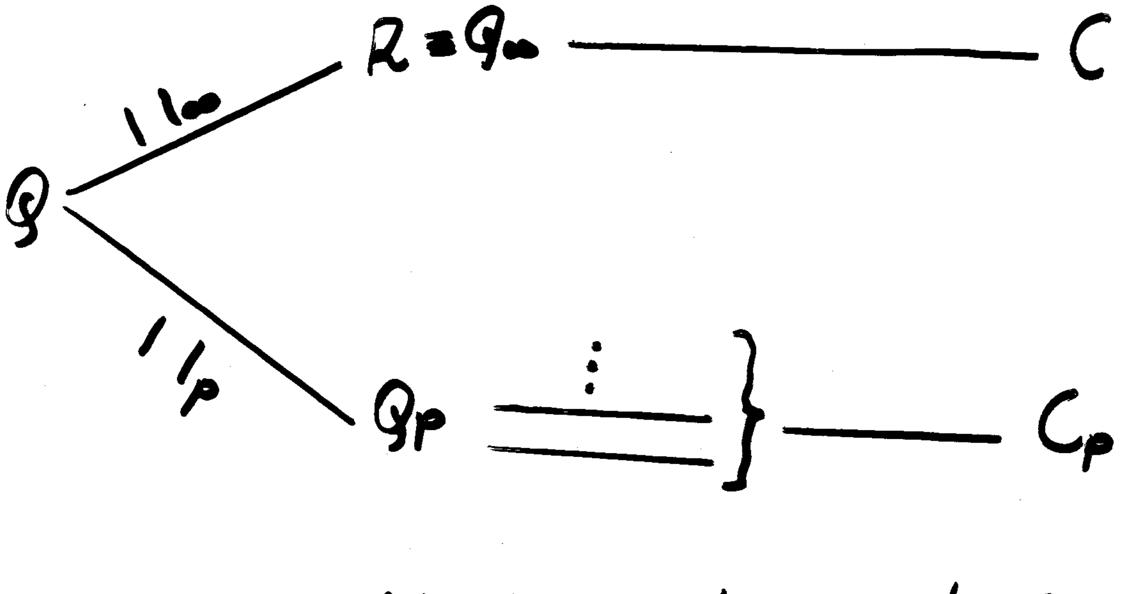
$$|x+y|_\infty \le |x|_\infty + |y|_\infty$$

Archimedean norm

$$|x+y|_p \le \max\{|x|_p, |y|_p\}$$

non-Archimedean (ultrametric) norm





$\mathbb{Q}_p$ = field of p-adic numbers

$p = 2, 3, 5, 7, 11, \ldots$

Geometries

$d(x,y) = |x-y|$

archimedean geom.

$d_\infty(x,y) = |x-y|_\infty$

$d_p(x,y) = |x-y|_p$

non-archimedean geom.

$$d_\infty(x,y) \le d_\infty(x,z) + d_\infty(z,y)$$

$$d_p(x,y) \le \max\{d_p(x,z),\ d_p(z,y)\}$$

p-adic spaces

closed ball $B_a(r)$

$$B_a(r) = \{x \in \mathbb{Q}_p : |x-a|_p \le r\}$$

open ball $B_a(r^-)$

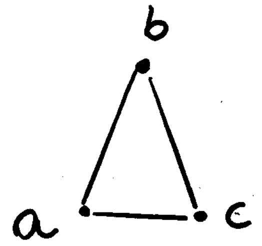
$$B_a(r^-) = \{x \in \mathbb{Q}_p : |x-a|_p < r\}$$

sphere $S_a(r)$

$$S_a(r) = \{x \in \mathbb{Q}_p : |x-a|_p = r\}$$

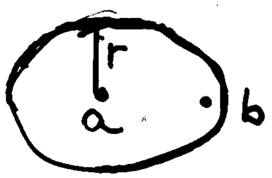
Some exotic properties of p-adic spaces

1) isosceles triangles



$$d_p(a,b) = d_p(b,c) \geq d_p(a,c)$$

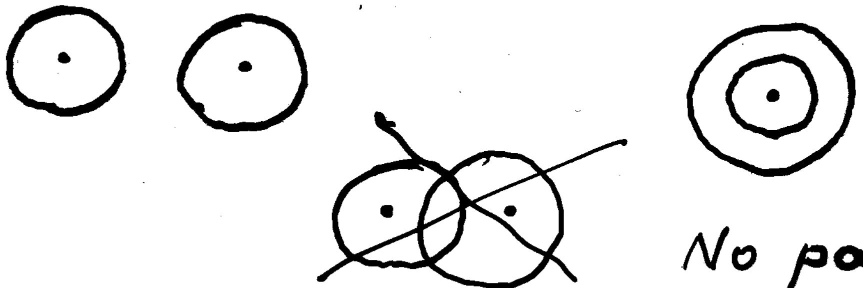
2)



$$B_a(r) = B_b(r)$$

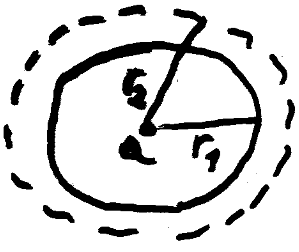
Each point of a ball may be its center!

3)



No partial intersection!

4)



$$|x-a|_p \leq p^\nu < p^{\nu+1}$$

closed = open

clopen sets

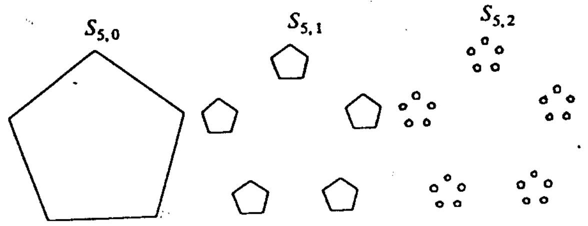


FIG. 4.

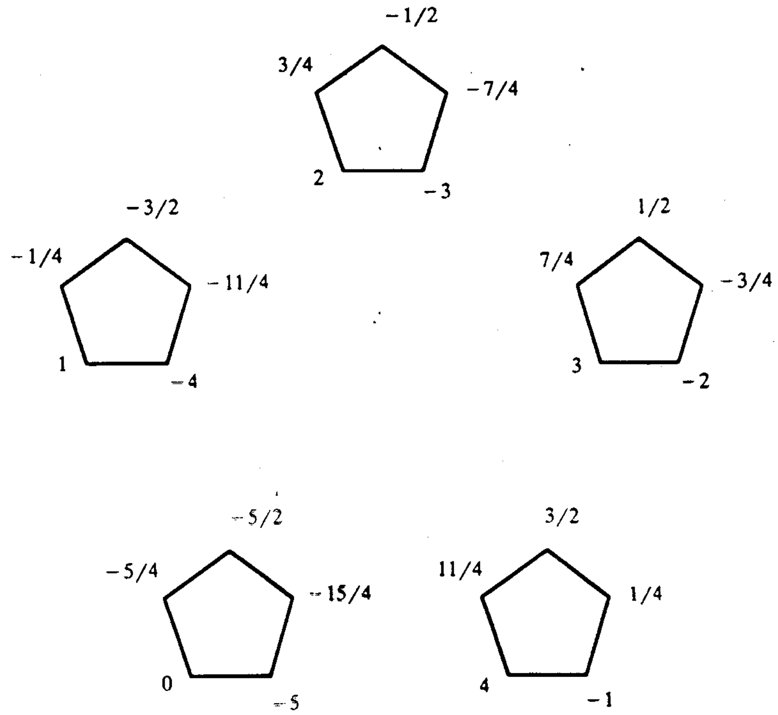


FIG. 7.

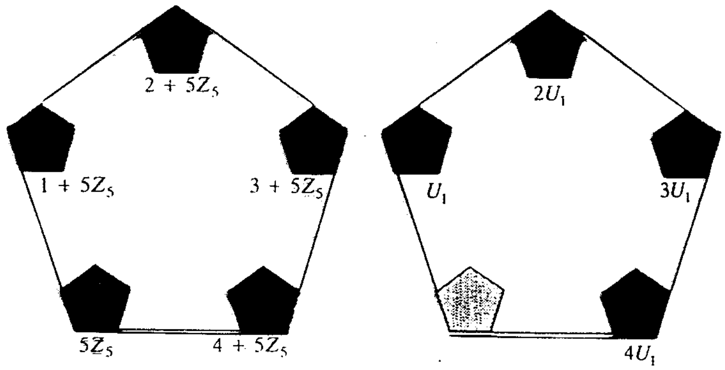


FIG. 9.

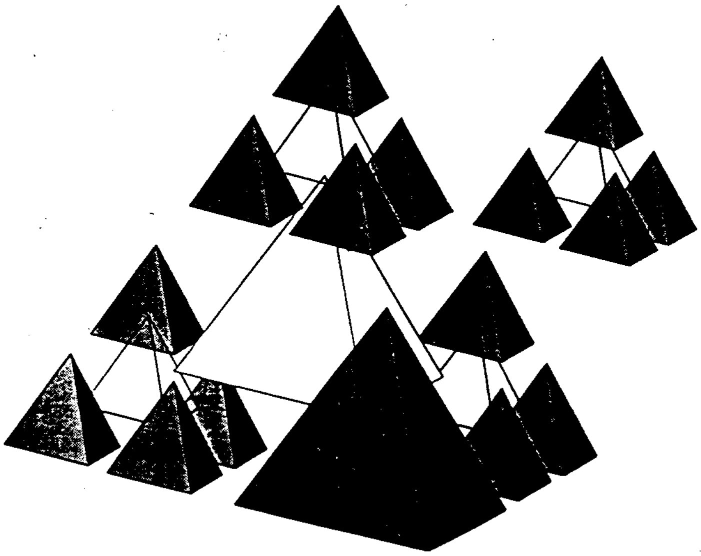


Figure 7: Space model of $\mathbf{Z}_5$

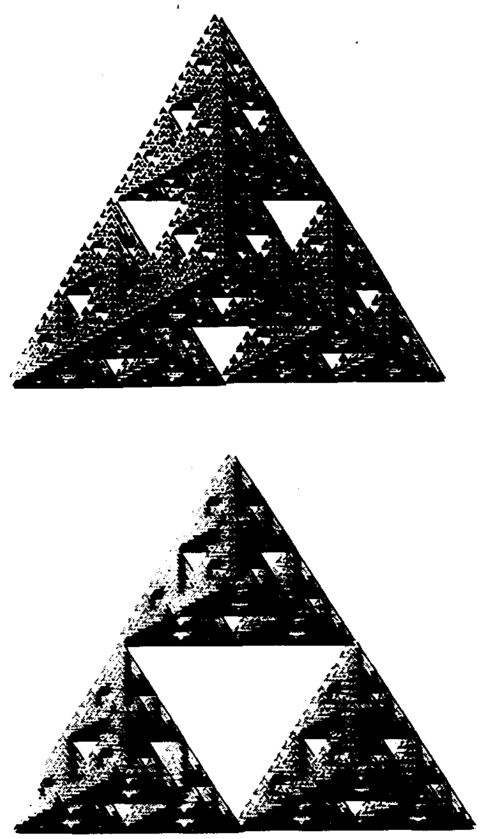


Figure 8: Top view of the space model of $\mathbf{Z}_5$

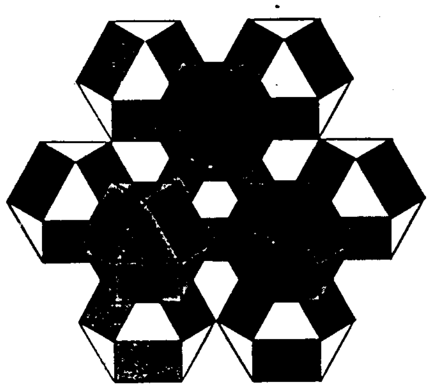


Figure 13: Model of $\mathbf{Z}_{13}$

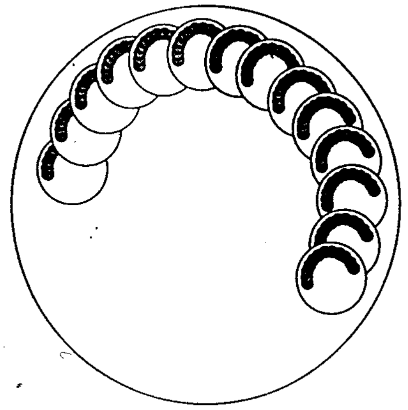
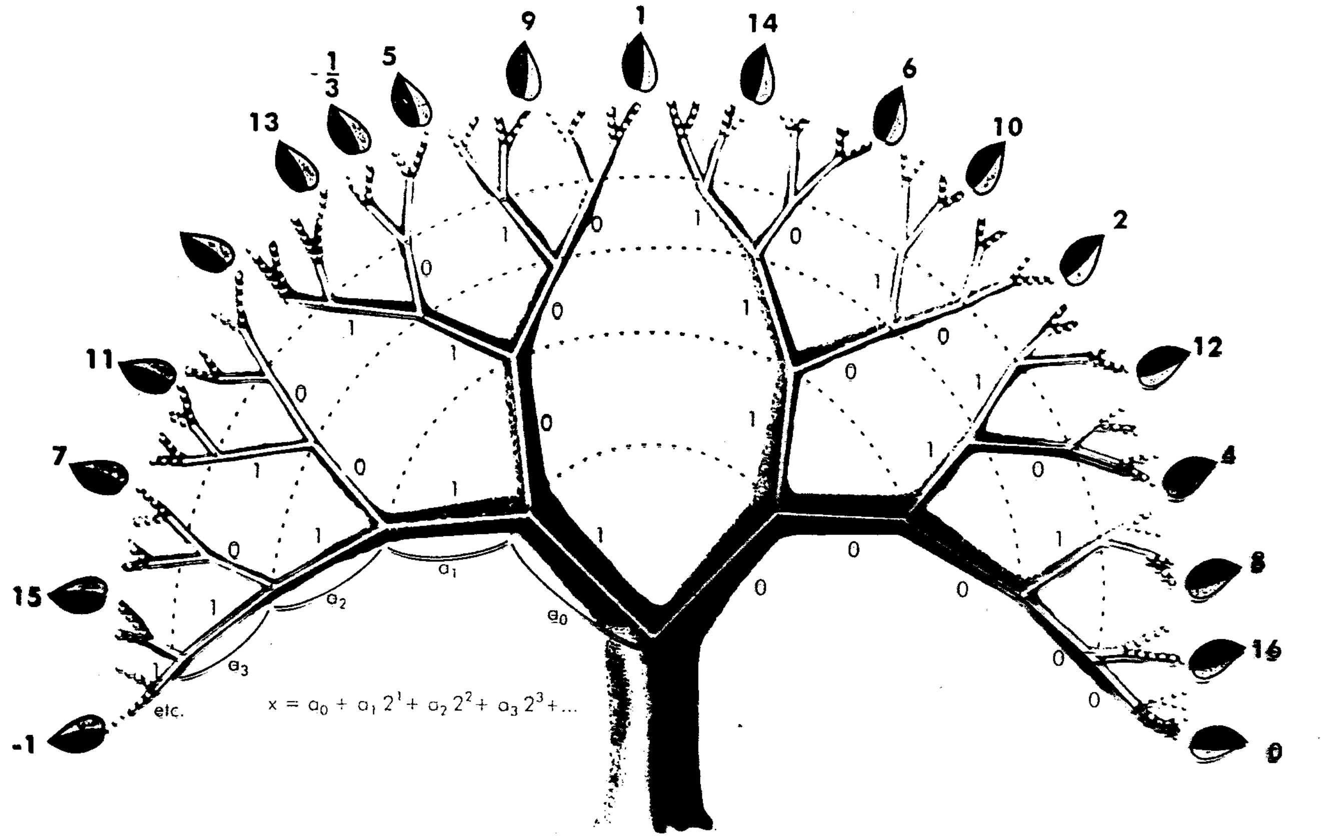


Figure 14: Antoine's model of $\mathbf{Z}_p$ ($p \succ 30$)



ADELES

$$a = (a_\infty, a_2, a_3, \dots, a_p, \cdots)$$

$a_\infty \in R$, $a_p \in Q_p$, $p \in S$

$a_p \in Z_p$, $p \notin S$ $\qquad S =$ finite set of p

$$\boxed{A = \bigcup_S A(S)}, \quad A(S) = R \times \prod_{p\in S} Q_p \times \prod_{p \notin S} Z_p .$$

$A =$ topological ring of adeles

$Z_p = \{x \in Q_p : |x|_p \le 1\}$

$r \in Q$

$a_r = (r, r, \dots, r, \dots)$ principal adele

ideles

$$A^\times = \bigcup_S A^\times(S), \quad A^\times(S) = R^\times \times \prod_{p\in S} Q_p^\times \times \prod_{p\notin S} Z_p^\times$$

$Z_p^\times = \{x \in Q_p : |x|_p = 1\}$

$Q_p^\times = Q_p \setminus \{0\}$

$A^\times =$ multiplicative group of ideles

adelic analysis

1. $A \longrightarrow A$

2. $A \longrightarrow C$

motivations

- All experimental data belong $\mathbb{Q}$
- $\mathbb{Q}$ is dense in $\mathbb{R}$, but also in $\mathbb{Q}_p$
- There is plausible analysis on $\mathbb{Q}_p$ as well as on $\mathbb{R}$
- General mathematical methods and fundamental physical laws should be invariant under $\mathbb{R} \leftrightarrow \mathbb{Q}_p$
- Is there any aspect of the Universe that cannot be described without use of p-adic numbers?
- There is

$$\Delta x \geqslant l_0 = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33} cm$$

- Also Hasse-Minkowski (local-global) principle
- Adelic method is a natural approach to investigate p-adic (non-archimedean) effects in physics
- String/M-theory and quantum cosmology allow p-adic and adelic generalization

classical analysis

$\mathbb{R} \longrightarrow \mathbb{R}$ $\qquad$ $\mathbb{R} \longrightarrow \mathbb{C}$

classical theory

p-adic analysis

$\mathbb{Q}_p \longrightarrow \mathbb{Q}_p$ $\qquad$ $\mathbb{Q}_p \longrightarrow \mathbb{C}$

quantum theory

- $\pi_p(x) = |x|_p^a$ multiplicative character
- $\chi_p(x) = \exp(2\pi i \{x\}_p)$ additive character

$$\pi_\infty(x) \prod_p \pi_p(x) = 1 \qquad x \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$$

$$\chi_\infty(x) \prod_p \chi_p(x) = 1 \qquad x \in \mathbb{Q}$$

p-Adic and adelic generalization of string amplitudes

Veneziano amplitude

$a = -\alpha(s) = -1 - \frac{1}{2}s$
$s + t + u = -8$
$a + b + c = 1$

$$A(k_1, \cdots, k_4) \equiv A_\infty(a,b)$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} + \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} + \frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)}$$ Volovich

$$= \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)}$$ Aref'eva, Volovich, B.D.

$$= \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx$$ Freund, ...

$$= g^2 \int \mathcal{D}X \exp\left(-\frac{2\pi i}{h} \frac{T}{2} \int d\sigma d\tau\, \partial^\alpha X^\mu \partial_\alpha X_\mu\right)$$
$$\times \int \prod_{j=1}^4 \exp\left(\frac{2\pi i}{h} k_\mu^{(j)} X^\mu(\sigma_j, \tau_j)\right) d\sigma_j$$ B.D.

$$A_p(a,b) = \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} dx$$

$A_p \in C$
$x \in Q_p$
$a, b, c \in R$

$$A_p(a,b) = \frac{1-p^{a-1}}{1-p^{-a}} \frac{1-p^{b-1}}{1-p^{-b}} \frac{1-p^{c-1}}{1-p^{-c}}$$

$$\prod_p A_p(a,b) = \frac{\zeta(a)}{\zeta(1-a)} \frac{\zeta(b)}{\zeta(1-b)} \frac{\zeta(c)}{\zeta(1-c)}$$

$$\boxed{A_\infty(a,b) \prod_p A_p(a,b) = 1}$$ Freund-Witten prod. formula

regularization
only for 4-point amplitudes

NEW APPROACH TO p-ADIC AND ADELIC STRING AMPLITUDES

B.D.
hep-th/0005200

general assumptions

(i) strings and space-time are adelic

(ii) adelic quantum theory with Feynman's path integral

(iii) ordinary theory is an effective limit of the adelic one

adelic string has real and p-adic properties

$$A_p(k_1,\dots,k_4) = g_p^2 \int \mathcal{D}X(\sigma,\tau)\, \chi_p\left(\frac{T}{2h}\int d^2\sigma\, \partial^\alpha X_\mu \partial_\alpha X^\mu\right) \times$$
$$\times \prod_{j=1}^{4} \int d^2\sigma_j\, \chi_p\left(-\frac{1}{h} k_\mu^{(j)} X^\mu(\sigma_j,\tau_j)\right)$$

(adelic): world sheet (σ,τ)
p-adic: Minkowski space (X^μ)
momenta (k_μ)

$A_p(k_1,\dots,k_4)$ is complex-valued

p-ADIC AND ADELIC QM

p-adic QM

Vladimirov, Volovich

$$\left(L_2(\mathbb{Q}_p),\ W_p(z_p),\ U(t_p)\right)$$

- $L_2(\mathbb{Q}_p)$ Hilbert space on $\mathbb{Q}_p$
- $W_p(z_p)$ Weyl quantization on $L_2(\mathbb{Q}_p)$
- $V(t_p)$ unitary repr. of evolution oper. on $L_2(\mathbb{Q}_p)$

$$U_p(t)\psi(x) = \int_{\mathbb{Q}_p} \mathcal{K}_p(x,t;y,0)\psi(y)\,dy$$

$$\mathcal{K}_p(x'',t'';x',t') = \int_{\mathbb{Q}_p} \chi_p\left(-\int_{t'}^{t''} L(q,\dot{q},t)\,dt\right)\prod_t dq(t)$$

quadratic Lagrangians

Djordjevic, B.D.

$$\mathcal{K}_{\substack{p\\ \infty}}(x'',t'';x',t') = \lambda_{\substack{p\\ \infty}}\left(-\frac{1}{2}\frac{\partial^2\bar{S}}{\partial x''\partial x'}\right)\left|\frac{\partial^2\bar{S}}{\partial x''\partial x'}\right|^{\frac{1}{2}}_{\substack{p\\ \infty}}\chi_{\substack{p\\ \infty}}\left(-\bar{S}(x'',t'';x',t')\right)$$

number field invariant $(\mathbb{R}\leftrightarrow\mathbb{Q}_p)$

adelic QM

B. D.

$$\left(L_2(\mathbb{A}),\ W(z),\ U(t)\right)$$

$$U(t)\psi(x) = \int_{\mathbb{A}} \mathcal{K}(x,t;y,0)\psi(y)\,dy$$

$$\mathcal{K}(x'',t'';x',t') = \mathcal{K}_\infty(x''_\infty,t''_\infty;x'_\infty,t'_\infty)\prod_p \mathcal{K}_p(x''_p,t''_p;x'_p,t'_p)$$

$$U(t)\psi(x) = \chi(Et)\psi(x)$$ spectral problem

p-ADIC AND ADELIC QUANTUM COSMOLOGY

Quantum cosmology is designed to describe the very early evolution of the Universe.

minisupespace models allow application of quantum mechanics

p-adic and adelic minisupespace quantum cosmology is an application of p-adic and adelic quantum mechanics to cosmological models

- de Sitter model in $D=4$ and $D=3$ dimens.
- model with homogeneous scaler field
- anisotropic Bianchi model with three scale factors
- some two-dimensional minisuperspace models

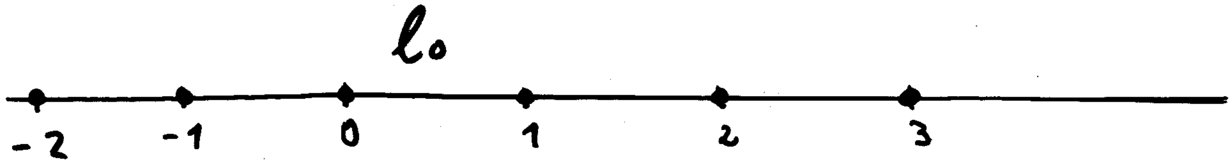
$$ds^2 = \sigma^2\left(-N^2(t)\,\frac{dt^2}{a^2(t)} + a^2(t)\,d\Omega_3^2\right)$$

adelic wave function of the Universe

Lj. Nešić, B.D.

$$\psi(x) = \psi_\infty(x_\infty) \prod_{p \in S} \psi_p(x_p) \prod_{p \notin S} \Omega(|x_p|_p)$$

Interpretation $\qquad x = x_\infty = \ldots = x_p \in \mathbb{Q}$



discreteness of space and time at the Planck scale which is a result of p-adic effects and adelic approach

$$\Omega(|x_p|_p) = \begin{cases} 1, & |x_p|_p \leq 1 \\ 0, & |x_p|_p > 0 \end{cases}$$

CONCLUSION

- strings, as well as the universe as a whole, can be regarded as adelic objects.
- Adelic approach is not alternative but more complete with respect to standard one.
- p-Adic and adelic quantum mechanics are formulated, as the first step to adelic string/M-theory.
- p-Adic and adelic generalization of the Feynman path integral method is very successful and promising.
- At the Planck scale p-adic quantum effects lead to some space-time discreteness which depend on adelic quantum state.

Basic Literature

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dragovich math-ph/..., hep-th/...

watkins http://...

http://www.maths.ex.ac.uk/~mwatkins/zeta/physics*.htm