MATH - PHYS II"

KOPAONIK, 1 9 - 12 9 2002

INTEGRABILITY
AND
RIGID - BODY SYSTEMS

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PLAN:

① INTRODUCTION TO INTEGRABILITY: CLASSICAL AND MODERN

② RIGID BODY SYSTEMS: BASIC NOTIONS

③ CLASSICAL INTEGRABLE RIGID BODY SYSTEMS

④ SOME RECENT RESULTS

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- V. DRAGOVIĆ, B. GAJIĆ THE LAGRANGE BITOP ON SO(4) × SO(4) AND GEOMETRY OF THE PRYM VARIETIES // SISSA PREPRINT 79/2001/FM math-ph/0201036

① INTEGRABILITY OF HAMILTONIAN SYSTEMS

$\underline{N^{2n}}\ (=\underline{T^*M})$, $H \in C^\infty(N)$ – **HAMILTONIAN**

(IN NATURALE SYSTEMS $\underline{H = T+U}$)

POISSON BRACKET: $\{,\}: C^\infty(N) \times C^\infty(N) \to C^\infty(N)$

– LIE ALGEBRA + LEIBN. IDENTITY

EQUATIONS OF MOTION: $\boxed{\dot{x} = \{x, H\}}$ $\left(\text{locally: } \begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases}\right)$

FIRST INTEGRALS: $F: N \to \mathbb{R}$

- ANALITICALLY: $\frac{d}{dt}\left(F(x(t))\right) = 0$
- GEOMETRICALLY: V. FIELD TANGENT TO $F^{-1}(c)$
- ALGEBRAICALLY: $\{F, H\} = 0$

(specially: H is a first integral)

DEF. HAMILTONIAN SYSTEM (N^{2n}, H) IS COMPLETELY INTEGRABLE IF IT HAS n INDEPENDENT FIRST INTEGRALS $F_1, \ldots, F_n$ IN INVOLUTION:

$$\{F_i, F_j\} = 0 .$$

LIOUVILLE-ARNOL'D THEOREM

(N^{2n}, H) COMPLETELY INTEGRABLE

$$N_f = \{ x \in N \mid F_i(x) = f_i \}, \quad f = (f_1, \ldots, f_n)$$

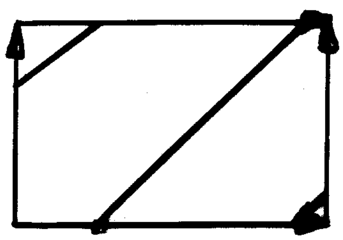
$\Rightarrow$ N_f invariant

if N_f compact, foliated on tori

locally, in a neighborhood of N_f a SYSTEM OF COORDINATES (φ, ω):

$$\boxed{\begin{aligned} \dot{\varphi}_i &= \omega_i \\ \dot{\omega}_i &= 0 \end{aligned}}$$

torus



integration in quadratures

action-angle variables (φ, ω) **implicit**

TWO MAIN PROBLEMS IN THE THEORY OF INT. SYSTEMS

① Is a given system integrable?

or

How to construct such system?

② If integrable, how to integrate it **explicitly**?

ALGEBRO-GEOMETRIC INTEGRATION

(FINITE-ZONE INTEGRATION)

MIDLE 70', MOSCOW: NOVIKOV, DUBROVIN, KRICHEVER

INFINITE-DIMENSIONAL HAMILTONIAN SYSTEMS FOR KdV, sin-GORDON, KP equations.

LAX REPRESENTATION: $\dot{L}(\lambda) = [\underset{\Updownarrow}{L(\lambda)}, A(\lambda)]$

$$\left(L(t) = U(t)\, L(0)\, U(t)^{-1} \right)$$

$\Rightarrow$ <u>SPECTRAL POLYNOMIAL</u> $P(\lambda,\mu) = \det(L(\lambda) - \mu E)$

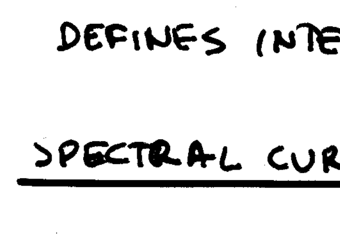
DEFINES INTEGRALS.

<u>SPECTRAL CURVE Γ</u>: $P(\lambda,\mu) = 0$

<u>BAKER-AKHIEZER FUNCTION</u> Ψ ON SPECTRAL CURVE Γ:

$$\begin{cases} L(\lambda(P))\,\Psi(P) = \mu(P)\,\Psi(P) \\ \partial_{\partial t}\Psi(P) = A(\lambda(P))\,\Psi(P) \end{cases}, \quad P \in \Gamma$$

$\mathcal{D}$ - pole divisor of Ψ



$$\xrightarrow{\text{ABEL MAP}} \quad T^2 = \mathbb{C}^2/\Lambda$$

torus $T^2 \cong \underline{Jac}(\Gamma)$

FINITE-ZONE INTEGRATION PROCEDURE:

$$\begin{tikzcd}
\text{initial data} \arrow[r, mapsto] & \dot{L}(0), A(0) \arrow[r, mapsto] & \Psi(0) \arrow[r, mapsto] & \boxed{\mathcal{D}(0)} \arrow[d, mapsto] \\
\text{solution of the system} & L(t), A(t) \arrow[l, mapsto] & \Psi(t) \arrow[l, mapsto] & \boxed{\mathcal{D}(t)} \arrow[l, mapsto]
\end{tikzcd}$$

RIGID BODY SYSTEMS

1) $$G = \iiint_\tau r \times (dm\, v) = \iiint_\tau r \times v \rho\, d\tau$$ ①

Кинетички моменат – kinetic momentum

$$\frac{dG}{dt} = \iiint_\tau (\dot{r} \times v + r \times \dot{v}) \rho\, d\tau$$

$$= \iiint_\tau r \times dF$$

} 2. Њутнов закон

NEWTON'S EQUAT.

$$L = \iiint r \times dF$$

главни моменат сила

$$\boxed{\frac{dG}{dt} = L}$$

PRINCIPLE FORCE MOMENT

Промена кинетичког момента = главном моменту сила

2) Кретање око непокретне тачке – ROTATIONS ABOUT FIXED POINT

$$\boxed{v = \Omega \times r}$$

$a \times (b \times c) = b(ac) - c(ab)$

$$G = \iiint_\tau [\Omega r^2 - r(r \cdot \Omega)]\, dm$$

$i \quad j \quad k$

$\Omega = (p, q, r)$

$r = (x, y, z)$

$$G = i \iiint_\tau \{p(y^2+z^2) - xyq - xzr\}\, dm$$

$$+ j \iiint_\tau \{q(z^2+x^2) - yzr - yxp\}\, dm$$

$$+ k \iiint_\tau \{r(x^2+y^2) - zxp - zyp\}\, dm$$

$$G = iAp + jBq + kCz$$

PRINCIPAL MOMENTA OF INERTIA

A, B, C ~~главни~~ ~~моменти~~ **инерције**

①

EULER EQUATIONS – I GROUP

Πρώτη ομάδα φόρμουλα Όιλεριαν

$$G = Ap\,i + Bq\,j + Cr\,k \quad , \quad \underline{\omega} = p\,i + q\,j + r\,k$$

$$\frac{dG}{dt} = A\dot{p}\,i + B\dot{q}\,j + C\dot{r}\,k + \begin{vmatrix} i & j & k \\ p & q & r \\ Ap & Bq & Cr \end{vmatrix}$$

$$= i\left[A\dot{p} + (C-B)qr\right] + j\left[B\dot{q} + (A-C)rp\right] + k\left[C\dot{r} + (B-A)pq\right.$$

$$L = \begin{vmatrix} i & j & k \\ x_0 & y_0 & z_0 \\ Mg\gamma' & Mg\gamma' & Mg\gamma'' \end{vmatrix} = i\,Mg(y_0\gamma'' - z_0\gamma') + j\,Mg(z_0\gamma - x_0\gamma'') + k\,Mg(x_0\gamma' - y_0\gamma)$$

$$A\dot{p} + (C-B)qr = Mg(y_0\gamma'' - z_0\gamma')$$

$$B\dot{q} + (A-C)rp = Mg(z_0\gamma - x_0\gamma'') \qquad (\text{I})$$

$$C\dot{r} + (B-A)pq = Mg(x_0\gamma' - y_0\gamma)$$

EULER EQUATIONS – I GROUP

Δεύτερη ομάδα φόρμουλα Όιλερια

$$\bar{k} = \gamma i + \gamma' j + \gamma'' k \qquad \frac{d\bar{k}}{dt} = 0 \Leftrightarrow i\dot{\gamma} + j\dot{\gamma}' + k\dot{\gamma}'' = -\underline{g}\times\bar{k}$$

$$\dot{\gamma} = r\gamma' - q\gamma''$$

$$\dot{\gamma}' = p\gamma'' - r\gamma \qquad (\text{II})$$

$$\dot{\gamma}'' = q\gamma - p\gamma'$$

Први интеграли – THE FIRST INTEGRALS

$$dT = Mg\, d\overline{z_0} \quad \Rightarrow \quad T = Mg\,\overline{z_0} + C$$

$$\frac{1}{2}\iiint \rho v^2 d\tau = \frac{1}{2}\iiint \rho\,[\underline{\omega} \times r]^2 d\tau$$

$$= \frac{1}{2}\left(Ap^2 + Bq^2 + Cr^2\right)$$

1) Интеграл енергије

$$\| \quad Mg\left(x_0\gamma + y_0\gamma' + z_0\gamma''\right) + C_1 \qquad (\underline{\text{III}})$$

2) $G = Ap\,i + Bq\,j + Cr\,k$

$$G \cdot \overline{\kappa} = (Ap\,i + Bq\,j + Cr\,k)(\gamma i + \gamma' j + \gamma'' k) =$$

$$= Ap\gamma + Bq\gamma' + Cr\gamma'' = C_2 \qquad (\underline{\text{IV}})$$

пројекција момента импулса

3) $\gamma^2 + \gamma'^2 + \gamma''^2 = 1 \qquad (\underline{\text{V}})$

$$\dot{M} = [M, \underline{\omega}] + [\Gamma, \chi]$$

$$\dot{\Gamma} = [\Gamma, \underline{\omega}]$$

$$M = I\underline{\omega} + \underline{\omega} I$$

INTEGRABLE CASES

Интеграбилни случајеви

EULER

I 1751. Ојлер $x_0=y_0=z_0=0$

$$\dot M=[M,\Omega]$$

$$\dot\Gamma=[\Gamma,\Omega]$$

$$Ap^2+Bq^2+Cr^2=h$$

$$Ap\gamma+Bq\gamma'+Cr\gamma''=k$$

$$\gamma^2+\gamma'^2+\gamma''^2=1$$

$$A^2p^2+B^2q^2+C^2r^2=\ell^2$$

LAGRANGE POISSON

II 1758. Лагранж (1813 Пуасон). $A=B$, $x_0=y_0=0$

$$\frac{A-C}{A}=m,\quad \frac{Mgz_0}{A}=1$$

$$\frac{dp}{dt}-mqr=-\gamma'$$

$$\frac{dq}{dt}+mrp=\gamma$$

$$\frac{dr}{dt}=0 \Rightarrow r=R$$

$$p^2+q^2-2\gamma''=h$$

$$p\gamma+q\gamma'-(m-1)R\gamma''=k$$

$$r=R$$

$$\gamma^2+\gamma'^2+\gamma''^2=1$$

$$p\dot q-q\dot p+mR(p^2+q^2)-(m-1)R\gamma''=k \iff \rho^2\frac{d\sigma}{dt}+mR\rho^2-(m-1)R\gamma''=$$

$$\dot p^2+\dot q^2+2mR(p\dot q-q\dot p)+m^2R^2(p^2+q^2)+\dot\gamma''^2=1 \quad /\cdot 4\rho^2$$

$$p=\rho\cos\sigma,\quad q=\rho\sin\sigma \qquad \rho^2-2\gamma''=h$$

$$\left(\frac{d\rho^2}{dt}\right)^2+4\left[\rho^2\frac{d\sigma}{dt}+mR\rho^2\right]^2+4\rho^2\gamma''^2-4\rho^2=0$$

$$\left(\frac{d\rho^2}{dt}\right)^2=-\rho^6+a\rho^4+\ell\rho^2+c$$

SUR LE PROBLÈME DE LA ROTATION D'UN CORPS SOLIDE AUTOUR D'UN POINT FIXE[1]

PAR

SOPHIE KOWALEVSKI

À STOCKHOLM.

§ 1.

Le problème de la rotation d'un corps solide pesant autour d'un point fixe peut se ramener, comme on sait, à l'intégration du système d'équations différentielles suivant:

$$(1)\qquad \begin{aligned} A\frac{dp}{dt} &= (B - C)qr + Mg(y_0\gamma'' - z_0\gamma'), & \frac{d\gamma}{dt} &= r\gamma' - q\gamma'', \\ B\frac{dq}{dt} &= (C - A)rp + Mg(z_0\gamma - x_0\gamma''), & \frac{d\gamma'}{dt} &= p\gamma'' - r\gamma, \\ C\frac{dr}{dt} &= (A - B)pq + Mg(x_0\gamma' - y_0\gamma), & \frac{d\gamma''}{dt} &= q\gamma - p\gamma'. \end{aligned}$$

Les constantes A, B, C, Mg, x_0, y_0, z_0 qui figurent dans ces équations ont la signification suivante.

A, B, C sont les axes principaux de l'ellipsoïde d'inertie du corps considéré, relativement au point fixe.

M est la masse du corps;

g l'intensité de la force de gravité;

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[1] Ce mémoire est le résumé d'un travail auquel l'Académie des Sciences de Paris, dans sa séance solennelle du 24 décembre 1888, a décerné le prix Bordin élevé de 3000 à 5000 francs.

x_0, y_0, z_0 les coordonnées du centre de gravité du corps considéré dans un système de coordonnées, dont l'origine est au point fixe et dont la direction coïncide avec celle des axes principaux de l'ellipsoïde d'inertie.

Jusqu'à présent on n'était parvenu à intégrer ces équations que dans deux cas particuliers:

1) Le cas de Poisson (ou d'Euler) où l'on a $x_0 = y_0 = z_0 = 0$,

2) Le cas de Lagrange où l'on a $A = B$, $x_0 = y_0 = 0$.

Dans ces deux cas l'intégration s'opère à l'aide des fonctions $\vartheta(u)$ dont l'argument est une fonction entière linéaire du temps.

Les six quantités p, q, r, γ, γ', γ'' sont dans ces deux cas des fonctions uniformes du temps, n'ayant d'autres singularités que des pôles pour toutes les valeurs finies de la variable.

Les intégrales des équations différentielles considérées conservent-elles cette propriété dans le cas général?

Si tel était le cas il faudrait pouvoir intégrer ces équations différentielles à l'aide de séries de la forme

$$(2)\qquad \begin{aligned} p &= t^{-n_1}(p_0 + p_1t + p_2t^2 + \ldots), \\ q &= t^{-n_2}(q_0 + q_1t + q_2t^2 + \ldots), \\ r &= t^{-n_3}(r_0 + r_1t + r_2t^2 + \ldots), \\ \gamma &= t^{-m_1}(f_0 + f_1t + f_2t^2 + \ldots), \\ \gamma' &= t^{-m_2}(g_0 + g_1t + g_2t^2 + \ldots), \\ \gamma'' &= t^{-m_3}(h_0 + h_1t + h_2t^2 + \ldots), \end{aligned}$$

où n_1, n_2, n_3, m_1, m_2, m_3 désignent des nombres entiers positifs, et ces séries, pour pouvoir représenter le système général d'intégrales des équations différentielles considérées, devraient contenir *cinq* constantes arbitraires.

Il faut donc examiner si une pareille intégration est possible.

On s'assure facilement, en comparant les exposants des premiers termes dans les membres gauches et dans les membres droits des équations considérées que l'on doit avoir

$$n_1 = n_2 = n_3 = 1, \qquad m_1 = m_2 = m_3 = 2.$$

KOVALEVSKAJA

3) Случај Ковалевске (1889) $A = B = 2C$, $y_0 = z_0 =$

$$2\dot p = qr$$

$$2\dot q = -pr - c\gamma''$$

$$\dot r = c\gamma'$$

$$c = \frac{Mg x_0}{C}$$

$$\left\{\begin{array}{l} 2(p^2+q^2)+r^2 = 2c\gamma + 6\ell_1 \\ 2(p\gamma + q\gamma') + r\gamma'' = 2\ell \\ \gamma^2+\gamma'^2+\gamma''^2 = 1 \end{array}\right.$$

$$[(p+qi)^2 + c(\gamma+i\gamma')]\,[(p-qi)^2 + c(\gamma-i\gamma')] = k^2$$

$$\Updownarrow$$

$$\boxed{(p^2+q^2-c\gamma)^2 + (2pq + c\gamma')^2 = k^2}$$

$$A > B > C$$

$$x_0\sqrt{A(C-B)} + y_0\sqrt{B(A-C)} + z_0\sqrt{C(B-A)} = 0$$

$$\Downarrow$$

$$\left.\begin{array}{l} x_0\sqrt{A(B-C)} + z_0\sqrt{C(A-B)} = 0 \\ y_0 = 0 \end{array}\right\}$$

HESS-APELROT CASE

случај Хес-Апелрота 1890-94

INTEGRABLE ON

интеграбилан на $\boxed{Ax_0 p + Cz_0 r = 0}$

THE EULER - POISSON EQUATIONS

on $so(n) \times so(n)$

(motion OF ~~A~~ HEAVY n-DIMENSIONAL RIGID BODY FIXED AT A POINT)

$$\dot{M} = [M, \Omega] + [\Gamma, X]$$

$$\dot{\Gamma} = [\Gamma, \Omega]$$

• $I = \mathrm{diag}(I_1, \ldots, I_n)$	mass distribution
• $\Omega \in so(n)$	angular velocity
• $M = I\Omega + \Omega I$	Kinetic momentum
• $X \in so(n)$	constant matrix (center of mass)
• $\Gamma \in so(n)$	

LAGRANGE BITOP

$$
\begin{aligned}
\dot{M} &= [M, \Omega] + [\Gamma, X] \\
\dot{\Gamma} &= [\Gamma, \Omega]
\end{aligned}
\qquad (1)
$$

DEF. $n=4$, $I_1 = I_2 = a$, $I_3 = I_4 = b$

$$X = \begin{bmatrix} \begin{matrix} 0 & X_{12} \\ -X_{12} & 0 \end{matrix} & \bigcirc \\ \bigcirc & \begin{matrix} 0 & X_{34} \\ -X_{34} & 0 \end{matrix} \end{bmatrix}$$

$a \neq b$, $X_{12}, X_{34} \neq 0$

$|X_{12}| \neq |X_{34}|$

DEFINES LAGRANGE BITOP.

Prop 1. $L(\lambda) = \lambda^2 C + \lambda M + \Gamma$, $C = (a+b)X$

$A(\lambda) = \lambda X + \Omega$

$$\frac{d}{dt} L(\lambda) \underset{\uparrow}{=} [L(\lambda), A(\lambda)] \iff (1)$$

LAX REPRESENTATION ⇒

PROP 2. LAGRANGE BITOP IS COMPLETELY INTEGRABLE in the LIOUVILLE SENCE.

RATIU (1982):

- GENERALIZED SYMMETRIC CASE

$$I_1 = \dots = I_n \, , \quad \chi \text{ arbitrary}$$

- GENERALIZED LAGRANGE CASE

$$I_1 = I_2 = a, \ I_3 = \dots = I_n = b, \quad \chi_{ij} = 0 \text{ if } (i,j) \notin \{(1,2),(2$$

$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma \qquad (*)$$

PROP 3 THE E-P EQS (1) HAVE THE LAX REPRESENTATION WITH THE L OPERATOR OF THE FORM (*) IF AND ONLY IF IT IS:

- GEN. SYMM. CASE
- GEN. LAGRANGE CASE
- LAGRANGE BITOP

CONTEREXAMPLE TO RATIU'S THEOREM - TROFIMOV, FOMENKO (1995) Theorem 15, ch 53.

	$n \geq 4$	
$n=3$	$n=4$	
1751 EULER $x_0 = y_0 = z_0 = 0$	MANAKOV 1976 L-A pair FRAMM 187?	
1788 LAGRANGE $A = B$, $x_0 = y_0 = 0$	LAGRANGE BITOP GAJIĆ, VD ~1999 L-A	GENERALISED LAGRANGE CASE ~1980 RATIU L-A
	GENERALIZED SYMMETRIC CASE ~1980 RATIU L-A	
1889 KOVALEVSKAYA $A = B = 2C$, $y_0 = z_0 = 0$	~1989 REIMAN, SEMENOV-TIEN SHANSKYI L-A	
~1890 HESS-APELROT GAJIC, VD ~1999 L-A		
~1900 GORYACHEV CHAPLYGIN SKLANIN, ... ~1989 L-A		

WHY IT IS CALLED LAGRANGE BITOP ?

$$so(4) \cong so(3) \oplus so(3)$$

$$\Downarrow$$

$$\dot p_1 - m q_1 r_2 = \gamma^1_1 \qquad \dot p_2 - m q_2 r_1 = \gamma^2_1$$

$$\dot q_1 + m p_1 r_2 = \gamma^1_2 \qquad \dot q_2 + m p_2 r_1 = \gamma^2_2$$

$$(a+b)\dot r_1 + (a-b)\dot r_2 = 0 \qquad (a-b)\dot r_1 + (a+b)\dot r_2 = 0$$

$$\Downarrow$$

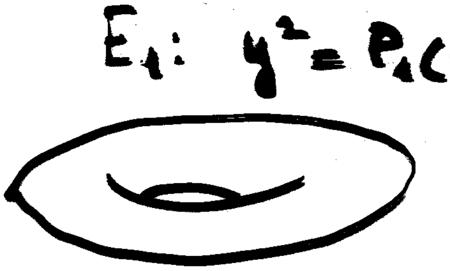
$$\vdots$$

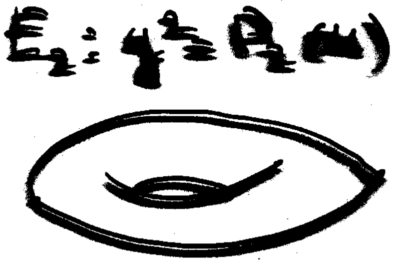
$$\dot u_1^2 = P_1(u_1) \qquad \dot u_2^2 = P_2(u_2)$$

$$\deg P_i = 3$$

two elliptic curves

$E_1 : y^2 = P_1(u)$ $\qquad$ $E_2 : y^2 = P_2(u)$





Algebro-geometric integration

NOVIKOV 1974 KdV eq.

DUBROVIN 1975

KRICHEVER 1976, 77 KP eq, BAKER-AKHIEZER FUN.

Matrix polynomial operators

DUBROVIN 1976

MUMFORD 1979
ADLER
VAN MOERBEKE

RIGID BODY SYSTEMS

MANAKOV 1976

RATIU 1980-82

$$L(\lambda) = \begin{bmatrix} * & 0 & * & * \\ 0 & * & * & * \\ * & * & * & 0 \\ * & * & 0 & * \end{bmatrix}$$

LAGRANGE
BITOP

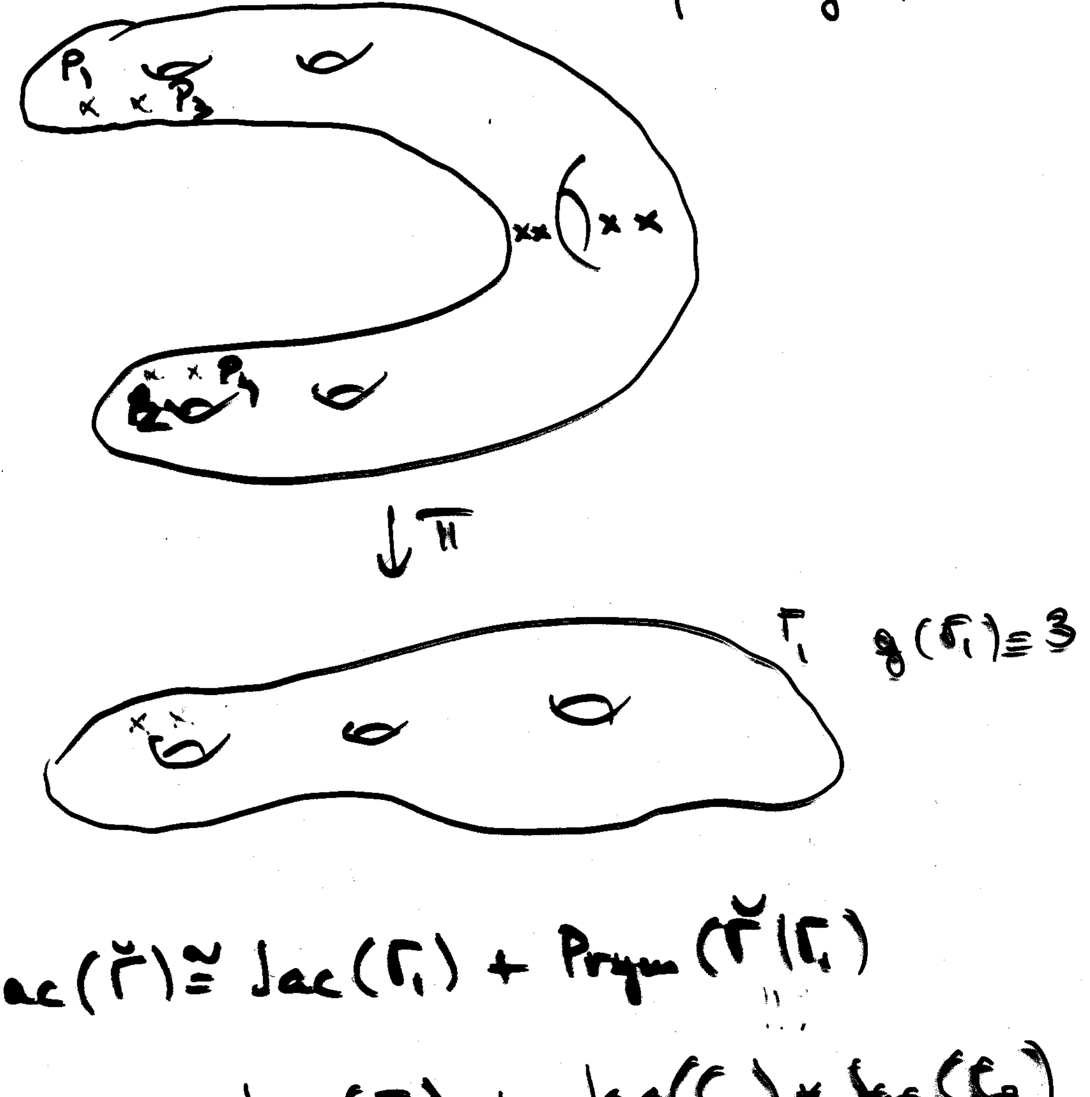
[illegible]

$$\Gamma: \; p(\lambda,\mu) := \mu^4 + P(\lambda)\mu^2 + Q^2(\lambda) = 0$$

$$\deg P = \deg Q = 4$$

[illegible] $\tau: \Gamma \to \Gamma \quad (\lambda,\mu) \mapsto (\lambda,-\mu)$

$$\Gamma_1 = \Gamma/\tau \quad : \quad \mu_1^2 = \frac{P^2(\lambda)}{4} - Q^2(\lambda)$$



$$\mathrm{Jac}(\check{\Gamma}) \cong \mathrm{Jac}(\Gamma_1) + \mathrm{Prym}(\check{\Gamma}|\Gamma_1)$$

$$\cong \mathrm{Jac}(\Gamma_1) + \mathrm{Jac}(C_1) \times \mathrm{Jac}(C_2)$$

$$C_{1,2}: \; \mu_1^2 = \frac{P}{2}(\lambda) \pm Q(\lambda)$$

$C_1: \mu_1^2 = P(\lambda)/2 + Q(\lambda)$; $C_2: \mu_1^2 = P(\lambda)/2 - Q(\lambda)$

Theorem 1. (i) $E_i = \mathrm{Jac}(C_i)$, $i = 1,2$

(ii) $\underset{\mathrm{Prym}(\tilde\Gamma|\Gamma_1)}{\underset{\parallel}{\Pi'}} \cong E_1 \times E_2$

⌜ Mumford-Dalalian theory (1974)

description of Prym varieties $\Pi = \mathrm{Prym}(\tilde\Gamma|\Gamma_1)$
where

(i) $\pi: \check\Gamma \to \Gamma_1$ is double unramified covering

(ii) Γ_1 is hyper-elliptic:

$$y^2 = P_{2g+2}(x)$$

$S = \{x_1, \dots, x_{2g+2}\}$ - set of zeroes of $P_{2g+2}(x)$

Each decomposition

$\{A, S \setminus A\}$, $|A|$ even,

corresponds to some $\mathrm{Prym}(\tilde\Gamma|\Gamma_1)$. ⌟

$$v^i_j = \frac{\lambda_i\, \theta(A(P_i) - A(P_j) + tU + z_0)}{\lambda_j\, \theta(\quad)(\quad)} .$$

$(i,j) \in \{(1,2), (2,1), (3,4) (4,3)\}$

$$\theta(A(P_i) - A(P_j) + tU + z_0) = 0 \quad \forall t$$

$$\Downarrow$$

Prop 6. $T \subset \Theta_{\tilde{\Gamma}}$

crucial geometric fact:

Mumford (1974): $\pi : \tilde{\Gamma} \to \Gamma_1$
double unramified covering,
Γ_1 arbitrary
there excists Π^-_1 a translate of $\Pi = \mathrm{Prym}(\tilde{\Gamma} | \Gamma_1)$
such that

$$\Pi^- \subset \Theta_{\tilde{\Gamma}}$$

THE LAGRANGE BITOP HIERARCHY AND equally-splitted double hyper-elliptic coverings

$$A(\lambda) = \lambda X + \Omega$$

HIERARCHY: $L_B^{(N)}(\lambda) = \lambda^N B + \lambda^{N-1} M_1 + \dots + M_N$

$N \geq 2, \quad B = d X$

$$\dot{L}_B^{(N)} = [L_B^{(N)}, A] \iff \frac{\partial M_N}{\partial t} = [M_N, \Omega]$$

$$\frac{\partial M_k}{\partial t} + [X, M_{k+1}] = [M_k, \Omega]$$

$$[X, M_1] = [B, \Omega]$$

$\Gamma_N: \; p_N(\lambda, \mu) = \mu^4 + P_N(\lambda)\mu^2 + [Q_N(\lambda)]^2 = 0$

$\deg P_N = \deg Q_N = 2N$

$\pi: \check{\Gamma}_N \to \Gamma_N'$ double unramified covering over

$\Gamma_N' : \; \mu_1^2 = P_N^2(\lambda)/4 - Q_N^2(\lambda)$

hyper-elliptic, $g_N = 2N-1$

Mumford-Dalalian: π corresponds to the division of the set of zeroes on subsets defined by $P_N/2 - Q_N$.

Theorem Lagrange bitop hierarchy realizes all such coverings.