The one-loop effective action for QED on NC space

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1) NC QED
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① Noncommutative QED

- In order to consider noncommutative field theories one can start with an abstract algebra: generators, x^μ + relations, e.g. $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ *
  or $[x^\mu, x^\nu] = f^{\mu\nu}{}_\rho x^\rho$, ...

  Define fields via transformation properties under some transformations, e.g. (infinitesimal) gauge transformations $\delta_\alpha \psi = i\alpha\psi$
  with $\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = \delta_{\alpha\times\beta}$

- One can also consider a representation of NC algebra by, e.g. fields on $\mathbb{R}^4$ with a given (noncommutative) multiplication.
  canonical structure * $\leftrightarrow$ Moyal product:

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)\, g(y)\Big|_{y\to x}$$

NC $U(1)$:

- group element $\hat{U}(x) = e^{i\hat{\theta}(x)}\Big|_*$ (by Taylor expansion)
- vector potential $\hat{A}_\mu$

$$\hat{A}'_\mu = \hat{U} * \hat{A}_\mu * \hat{U}^{-1} + i\,\hat{U} * \partial_\mu \hat{U}^{-1}$$

- field strength $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i\,[\hat{A}_\mu \overset{*}{,} \hat{A}_\nu]$

$$\hat{F}'_{\mu\nu} = \hat{U} * \hat{F}_{\mu\nu} * U^{-1}$$

- matter $\hat{\Psi}$, $\hat{\Psi}' = \hat{U} * \hat{\Psi}$

$$\hat{D}_\mu \hat{\Psi} = \partial_\mu \hat{\Psi} - i\,\hat{A}_\mu * \hat{\Psi}, \quad (\hat{D}_\mu \hat{\Psi})' = \hat{U} * D_\mu \hat{\Psi}$$

- action etc.

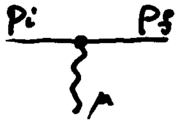
$$S = \int \left(-\frac{1}{4g^2} \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} + \hat{\bar{\Psi}} * i\hat{\not{D}}\hat{\Psi} - m\hat{\bar{\Psi}} * \Psi \right)$$

Quantization

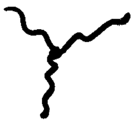
$$S = \int -\frac{1}{4}\hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} + \hat{\bar{\psi}} * i\hat{\not{D}}\hat{\psi} - m\hat{\bar{\psi}} * \psi$$
$$- \frac{1}{2\alpha}\partial_\mu \hat{A}^\mu * \partial_\nu \hat{A}^\nu + \frac{1}{2}\left(i\hat{\bar{c}} * \partial^\mu \hat{D}_\mu \hat{c} - i\partial^\mu \hat{D}_\mu \hat{c} * \bar{c} \right)$$

- propagators are the same as in the commutative $U(1)$ (θ-terms treated as interaction $\leftrightarrow$ θ = external field)

- Feynman rules

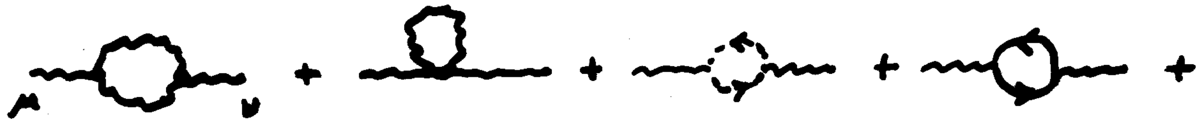


$$i\gamma^\mu e^{\frac{i}{2} p_i \theta p_f} = i\gamma^\mu e^{\frac{i}{2} p_i \tilde{p}_f} \qquad \tilde{p}^\mu = \theta^{\mu\nu} p_\nu$$



$\vdots$

E.g. corrections to the photon self-energy



$$i\Pi_f^{\mu\nu} = -4\int \frac{d^4 l}{(2\pi)^4} \frac{\mathrm{Tr}\, \gamma^\mu(\not{l} - \not{p})\gamma^\nu \not{l}}{(l-p)^2 l^2} \frac{1}{2}\left(1 - \cos(\tilde{p} l)\right)$$

$$\underset{\text{nonplanar}}{=} 4\int \frac{d^4 l}{(2\pi)^4} \frac{2l^\mu l^\nu - g^{\mu\nu} l^2}{l^4} e^{i\tilde{p} l}$$

$$= -16 i \alpha \frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^4}$$

UV/IR mixing
nonperturbative in θ
(integral is finite for large l, but the result diverges for $\tilde{p}$ small)

② SW map and θ-expansion

Seiberg and Witten showed that one can expand noncommutative fields $\hat{A}_\mu$, $\hat{F}_{\mu\nu}$, $\hat{\psi}$ in θ; the coefficients in the expansion depend on the corresponding fields for commutative U(1) or QED:

$$\hat{\lambda} = \lambda - \frac{1}{2}\theta^{\rho\sigma} A_\rho \partial_\sigma \lambda + O(\theta^2) \qquad \text{gauge par.}$$

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\theta^{\rho\sigma} A_\rho (\partial_\sigma A_\mu + F_{\sigma\mu}) + \cdots$$

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\rho\sigma}(F_{\mu\rho}F_{\nu\sigma} - A_\rho \partial_\sigma F_{\mu\nu}) + \cdots$$

$$\hat{\psi} = \psi - \frac{1}{2}\theta^{\rho\sigma} A_\rho \partial_\sigma \psi + \cdots$$

$\theta = 0$ - commutative limit.

SW map is not unique!

Another possible approach to NC U(1) and QED: expand the action in θ, then quantize.
In the outcome one gets the effects of noncomm. (in the Lagrangian) in orders of θ - when small.
But also: possibility to discuss renormalizability.

$$S = \int \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$-\frac{1}{2}\theta^{\rho\sigma}\int F_{\mu\rho}F_{\nu\sigma}F^{\mu\nu} - \frac{1}{4}F_{\rho\sigma}F_{\mu\nu}F^{\mu\nu}$$

$$+\frac{1}{2}\theta^{\rho\sigma}\int -i F_{\mu\rho}\bar{\psi}\gamma^\mu D_\sigma\psi + \frac{1}{2}F_{\rho\sigma}\bar{\psi}(-i\gamma^\mu D_\mu + m)\psi$$

$$+ \theta^2$$

$D_\mu\psi = \partial_\mu\psi - iA_\mu\psi$ ordinary covar. der.

③ Background field method

Set of fields ϕ with the classical action $S[\phi]$ and the sources J

Generating functional for the connected green funct. is

$$e^{W[J]} = \int D\phi \, e^{iS[\phi] + i\int J\phi}$$

Effective action $\Gamma[\phi_0] = W[J] - \int J\phi_0$ can be calculated by the saddle-point method after the expansion of action around the classical configuration ϕ_0,

$\phi = \phi_0 + \Phi$ and $J = -\frac{\delta S}{\delta \phi_0}$:

$$e^{i\Gamma[\phi_0]} \approx e^{iS[\phi_0]} \cdot \int D\Phi \, e^{\frac{i}{2}\int \Phi S^{(2)} \Phi}$$

where $S^{(2)} = \frac{\delta^2 S}{\delta \phi^2}\Big|_{\phi=\phi_0}$. Functional integration gives

$$\Gamma[\phi_0] = S[\phi_0] - \frac{1}{2i} \, STr \log S^{(2)}[\phi_0]$$

$\uparrow$ real ϕ !

Our case : A_μ real bosonic

ψ Dirac spinor = complex grassmann

$\Rightarrow$ one should use Majorana spinors $\psi = \psi_1 + i\psi_2$

$\rightarrow$ $\phi = (A_\mu, \psi_1, \psi_2)$

Action in Majorana spinors

$$S = \int \bar{\Psi}_1 (i\not{\partial} - m)\Psi_1 + \bar{\Psi}_2 (i\not{\partial} - m)\Psi_2 + i\bar{\Psi}_1 \not{A}\Psi_2 - i\bar{\Psi}_2 \not{A}\Psi_1 - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

$$-\frac{1}{2}\theta^{\varsigma\sigma}\int -\frac{i}{4}\Delta^{\alpha\beta\gamma}_{\sigma\varsigma\mu}\,\bar{\Psi}_1\,\gamma^\mu F_{\gamma\beta}\partial_\alpha \Psi_1 + \frac{1}{2} m F_{\varsigma\sigma}\bar{\Psi}_1\Psi_1 + 1\leftrightarrow 2$$

$$-\frac{i}{4}\Delta^{\alpha\beta\gamma}_{\sigma\varsigma\mu}\,\bar{\Psi}_1\,\gamma^\mu F_{\gamma\beta} A_\alpha \Psi_2 - 1\leftrightarrow 2$$

where $\Delta^{\alpha\beta\gamma}_{\sigma\varsigma\mu} = \delta^\alpha_\sigma \delta^\beta_\varsigma \delta^\gamma_\mu - \delta^\alpha_\varsigma \delta^\beta_\gamma \delta^\gamma_\mu$ + cyclic $\alpha\beta\gamma$

From it we can obtain its quadratic part:

$$S^{(2)} = \int (A_\alpha\ \bar{\Psi}_1\ \bar{\Psi}_2)\, B \begin{pmatrix} A_\beta \\ \Psi_1 \\ \Psi_2 \end{pmatrix}$$

<u>Structure</u> of B

$$B = \left(\begin{array}{c|cc} B_{11} & B_{12} & B_{13} \\ \hline B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{array}\right) \qquad STr\, B = Tr\, B_{11} - Tr\, B_{22} - Tr\, B_{33}$$

$$= \begin{pmatrix} \frac{1}{2} g^{\alpha\beta}\Box & 0 & 0 \\ 0 & i\not{\partial} & 0 \\ 0 & 0 & i\not{\partial} \end{pmatrix} + M$$

In order to expand log B around identity (perturbative expansion) we need I-matrices on diagonal

$$\Gamma = \frac{i}{2} STr \log B = \frac{i}{2} STr \log BC + \frac{i}{2} STr \log C^{-1}$$

$$= \frac{i}{2} STr \log (I + \Box^{-1} M C) + \frac{i}{2} STr \log C^{-1}\Box$$

infinite renormalization

$$C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -i\not{\partial} & 0 \\ 0 & 0 & -i\not{\partial} \end{pmatrix}$$

Finally,

$$\Gamma = S_0 + \frac{i}{2}\,\mathrm{STr}\log\left(I + \Box^{-1}N_0 + \Box^{-1}N_1 + \Box^{-1}T_1 + \Box^{-1}T_2\right)$$

$$= S_0 + \frac{i}{2}\sum_1^{\infty}\frac{(-)^{n+1}}{n}\,\mathrm{STr}\left(\Box^{-1}N_0 + \Box^{-1}N_1 + \Box^{-1}T_1 + \Box^{-1}T_2\right)^n$$

with

$$N_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & im\not\partial & 0 \\ 0 & 0 & im\not\partial \end{pmatrix} \qquad \text{0 fields}$$

$$N_1 = \begin{pmatrix} 0 & -i\bar{\Psi}\gamma^{\alpha}\not\partial & \bar{\Psi}\gamma^{\alpha}\not\partial \\ 2\gamma^{\beta}\psi & 0 & \not{A}\not\partial \\ -2i\gamma^{\beta}\psi & -\not{A}\not\partial & 0 \end{pmatrix} \qquad \text{1 field}$$

$$T_1 = \begin{pmatrix} -\partial_\mu V^{\mu\xi,\nu\sigma}\partial_\nu & \frac{i}{4}\theta^{\rho\sigma}\Delta^{\alpha\beta\gamma}_{\sigma\rho\mu}\partial_\gamma\left(-\partial_\beta\psi_1 i\gamma^{\mu} - \frac{m}{2}\delta^{\mu}_{\beta}\psi_1\right)\not\partial & \cdot \\ \cdot & -\frac{i}{4}\theta^{\rho\sigma}\left(-\frac{i}{2}\Delta^{\alpha\beta\gamma}_{\sigma\rho\mu}\gamma^{\mu}F_{\gamma\beta}\partial_\alpha + mF_{\rho\sigma}\right)\not\partial & 0 \\ \cdot & 0 & \cdot \end{pmatrix}$$

θ-linear, 1 field

$$T_2 = \theta^{\rho\sigma}\Delta^{\alpha\beta\gamma}_{\sigma\rho\mu}\begin{pmatrix} -\frac{1}{2}\bar{\Psi}\gamma^{\mu}\psi\partial_\gamma & \frac{i}{4}\left(\frac{1}{2}F_{\gamma\beta} + \partial_\gamma A_\beta\right)\bar{\Psi}\gamma^{\mu}\not\partial & \cdot \\ \cdot & \frac{i}{8}A_\alpha F_{\gamma\beta}\gamma^{\mu}\not\partial & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

θ-linear, 2 fields

$$V^{\mu\xi,\nu\sigma} = \frac{1}{2}\left(g^{\mu\nu}g^{\xi\sigma} - g^{\mu\sigma}g^{\xi\nu}\right)\theta^{\alpha\beta}F_{\alpha\beta}$$
$$+ g^{\mu\nu}\left(\theta^{\xi\alpha}F^{\sigma}{}_{\alpha} + \theta^{\sigma\alpha}F^{\xi}{}_{\alpha}\right) + \ldots$$
$$+ \theta^{\mu\xi}F^{\nu\sigma} + \theta^{\nu\sigma}F^{\mu\xi} + \ldots$$

④ Effective action

We calculate the divergent part of the effective action (in order to discuss renormalizability), but

- linear and second order in θ
- second order in fields (corrections to propagators)

Because of the term $\square^{-1} N_0$ in the expansion, which is 0-th order in θ and has no fields, it seems that there are contributions of all orders n.

However, only finitely many diverge! :

$$\left.\begin{array}{ll} (\square^{-1} N_0)^k \, \square^{-1} T_2 \,, & k = 1,2,3,4 \\ (\square^{-1} N_0)^k \, \square^{-1} N_1 \, \square^{-1} T_1 \,, & k = 1,2,3 \end{array}\right\} \theta\text{-linear}$$

$$(\square^{-1} N_0)^k \, (\square^{-1} T_1)^2 \,, \quad k = 1,2,3,4 \qquad \theta\text{-quadratic}$$

To calculate STr : momentum representation →

→ dimensional regularization → coordinate repr. →

→ traces and identities →

Eg.

$$\Gamma_b = -\frac{i}{4} \int dx\, dy \; g^{\nu\gamma} g^{\mu\sigma} \, V^{\alpha\beta}_{\nu\mu}(x) \, V^{\rho\delta}_{\sigma\gamma}(y) \, \partial^x_\alpha \partial^y_\delta G(y-x) \, \partial^x_\beta \partial^y_\rho G(x-y)$$

$$= \ldots$$

$$= \frac{1}{4(4\pi)^2 \varepsilon} \int dx \Big(\square \tilde{F}^{\mu\nu} \square \tilde{F}_{\mu\nu} + \frac{2}{15} \square \tilde{F}^{\mu\beta} \partial_\mu \partial^\nu \tilde{F}_{\nu\beta} + \frac{1}{5} \square F^{\mu\nu} \tilde{\partial}_\alpha \partial^\alpha F_{\mu\nu} - \frac{1}{4} \theta^2 \square F^{\mu\nu} \square F_{\mu\nu} \Big)$$

Bichl, Grimstrup, Popp, Schweda, Wulkenhaar

$$\tilde{F}^{\mu\nu} = \theta^\mu{}_\alpha F^{\alpha\nu} \,, \quad \tilde{\tilde{F}} = \theta_{\mu\nu} F^{\mu\nu} \,, \quad \tilde{\partial}^\mu = \theta^\mu{}_\alpha \partial^\alpha \,, \quad \theta^2 = \theta^{\rho\sigma} \theta_{\rho\sigma}$$

Full result

$$\Gamma_1 = \frac{1}{(4\pi)^2\varepsilon} \int dx \Big(4i\bar\Psi \not{\partial} \Psi - 16 m \bar\Psi\Psi - \frac{2}{3} F_{\mu\nu}F^{\mu\nu}$$

$$+ \theta^{\alpha\beta} \Big(\frac{1}{3} \bar\Psi \gamma_\alpha \partial_\beta \Box \Psi - \frac{im}{3} \bar\Psi \gamma_\alpha \partial_\beta \not{\partial} \Psi + \frac{1}{6} \bar\Psi \sigma_{\alpha\beta} \Box (i\not{\partial} - m)\Psi$$

$$+ m^2 \bar\Psi \gamma_\alpha \partial_\beta \Psi + \frac{m^2}{2} \bar\Psi \sigma_{\alpha\beta} \Box (i\not{\partial} - m)\Psi \Big)$$

$$- \frac{1}{120} \tilde F^{\rho\sigma} \Box^2 \tilde F_{\sigma\rho} + \frac{1}{30} \tilde F^{\rho\sigma} \Box^2 \tilde F_{\rho\sigma} - \frac{1}{30} \tilde F^{\rho\sigma} \Box \partial_\sigma \partial^\nu \tilde F_{\rho\nu}$$

$$+ \frac{m^2}{6} \tilde F^{\rho\sigma} \Box \tilde F_{\rho\sigma} - \frac{m^2}{12} \tilde F^{\rho\sigma} \Box \tilde F_{\sigma\rho} - \frac{m^2}{6} \tilde F^{\rho\mu} \partial_\mu \partial^\nu \tilde F_{\rho\nu} - \frac{m^2}{4} \tilde F_{\rho\sigma} \tilde F^{\sigma\rho}$$

$$+ \frac{i}{48} \theta^2 \bar\Psi \Box^2 \not{\partial} \Psi - \frac{i}{24} \theta^{\alpha\mu} \theta^\beta{}_\mu \bar\Psi \Box \not{\partial} \partial_\alpha \partial_\beta \Psi - \frac{i}{12} \theta^{\alpha\mu} \theta^\beta{}_\mu \bar\Psi \Box^2 \gamma_\alpha \partial_\beta \Psi$$

$$+ \frac{m}{12} \theta^{\alpha\mu} \theta^\beta{}_\mu \bar\Psi \partial_\rho \partial_\sigma \Box \Psi$$

$$+ \frac{5im^2}{48} \theta^2 \bar\Psi \Box \not{\partial} \Psi - \frac{im^2}{24} \theta^{\alpha\mu} \theta^\beta{}_\mu \bar\Psi \not{\partial} \partial_\alpha \partial_\beta \Psi - \frac{7im^2}{24} \theta^{\alpha\mu} \theta^\beta{}_\mu \bar\Psi \gamma_\alpha \partial_\beta \Box \Psi$$

$$+ \frac{m^3}{4} \theta^{\alpha\mu} \theta^\beta{}_\mu \bar\Psi \partial_\alpha \partial_\beta \Psi$$

$$+ \frac{5im^4}{24} \theta^2 \bar\Psi \not{\partial} \Psi - \frac{im^4}{3} \theta^{\alpha\mu} \theta^\beta{}_\mu \bar\Psi \gamma_\alpha \partial_\beta \Psi$$

$$- \frac{m^5}{8} \theta^2 \bar\Psi \Psi \Big)$$

$$+ \Gamma_b$$

⑤ Discussion

How can one expect to conclude something about renormalizability from θ-expansion, when

- θ is dimensionful parameter (of mass dim -2)
- considering only θ-linear (or quadratic) terms one looses information about the full structure of NC action

Idea is that the SW map, in this case its nonuniqueness, contains information about the complete theory

Namely, if the fields are expanded as

$$\hat{A}_\mu = \Sigma\, \theta^n A_\mu^{(n)} \quad , \quad \hat{\psi} = \Sigma\, \theta^n \psi^{(n)}$$

the fields with the following coefficients are also solutions of SW-equations

$$A_\mu^{(n)\prime} = A_\mu^{(n)} + \mathcal{A}_\mu^{(n)} \quad , \quad \psi^{(n)\prime} = \psi^{(n)} + \Psi^{(n)}$$

$\mathcal{A}_\mu^{(n)}, \Psi^{(n)}$ are gauge covariant expressions of appropriate dimension with <u>exactly</u> n-factors θ

These redefinitions produce in the action the following shifts:

$$\Delta S_b = \int dx \; (D_\nu F^{\mu\nu})\, \mathcal{A}_\mu^{(n)}$$

$$\Delta S_f = \int dx \; [\, \bar{\psi}(i\not{\partial}-m)\Psi^{(n)} + \bar{\Psi}^{(n)}(i\not{\partial}-m)\psi \,]$$

So:

if all divergent terms are of these types, we can redefine fields order by order in θ

NC $U(1)$

- $\Gamma_\theta = 0$, $\Gamma_{\theta^2} = \Gamma_b$ - allowed (Bichl, Grimstrup, Grosse, Popp, Schweda, Wulkenhaar)

- by counting dimensions one can see that in higher orders in θ divergent corrections of photon propagator will be of type allowed by redefinitions

NC QED

- θ^2 bosonic part $\quad m^2 \tilde{F}_{\rho\sigma} \tilde{F}^{\sigma\rho}$ - term is wrong
- θ fermionic part $\quad m \bar{\Psi} \gamma_\alpha \partial_\beta \not{\partial} \Psi$

$\Rightarrow$ only for $m=0$ propagators can be renormalized

- one can see that for $m=0$ also θ^2 fermionic part is OK.

- this is valid also for higher order corrections, as for $m=0$ they are

$$\underbrace{\theta \ldots \theta}_{n} A \underbrace{\partial \ldots \partial}_{k} A \qquad\qquad \underbrace{\theta \ldots \theta}_{n} \bar{\Psi} \gamma \ldots \underbrace{\partial \ldots \partial}_{k} \Psi$$

$$-2n + 1 + k + 1 = 4 \qquad\qquad -2n + \tfrac{3}{2} + k + \tfrac{3}{2} = 4$$

$$k = 2 + 2n \qquad\qquad k = 1 + 2n$$

- 4-fermion vertex! Wulkenhaar

$$S_{4\psi} = -\frac{1}{(4\pi)^2 \epsilon} \frac{1}{2} \theta^{\rho\sigma} \Delta^{\alpha\beta\gamma}_{\sigma\rho\mu} \int dx \; \bar{\Psi} \gamma^\mu \Psi \cdot \bar{\Psi} \gamma_\alpha \gamma_\beta \gamma_\gamma \Psi$$

(no derivative!