Introduction to

Open Bosonic SFT

Kopaonik 2002

I.
- Tachyon Condensation: Sen's conjectures
- Witten's SFT
- Various formulations
- Tachyon condensation revisited

II.
- Vacuum String Field Theory
- Classical Solutions: the sliver
- Fluctuation spectrum
- Lower dimensional lumps

III.
- Noncommutative solitons in FT
- Moyal formulation of VSFT
- SFT with background B field
- SFT Ancestors of nc solitons

REFERENCES

Recent reviews

K. Ohmori "A review on Tachyon Cond.." hepth/0102085

L. Rastelli, A. Sen, B. Zwiebach "Vacuum String Field Theory" hepth/0106010

I. Ya. Aref'eva et al. "Noncommutative FT and (Super) SFT" hepth/0111208

Tachyon condensation

A. Sen "Tachyon condensation on the Brane-Antibrane System" hepth/9805170

"Universality of the Tachyon Potential" hepth/9911116

Basic References

E. Witten "Non-commutative Geometry and String Field Theory" NPB 268 (1986) 253

D. Gross, A. Jevicki "Operator Formulation of Interacting String Field Theory" NPB 283 (1987) 1, NPB 287 (1987), 225

A. LeClair, M.E. Peskin and C.R. Preitschopf NPB 317 (1987) 411, NPB 317 (1989) 464

Vacuum String Field Theory

V. A. Kostelecky, R. Potting, "Analytical construction of a nonperturbative vacuum..." hepth/0008252

L. Rastelli, A. Sen, B. Zwiebach:
"String Field Theory around the tachyon vacuum" hepth/0012251
"Classical solutions in SFT..." hepth/0102112
"Half-strings, Projectors, ..." hepth/0105058
"Star algebra spectroscopy" hepth/0111281

D. Gaiotto, L. Rastelli, A. Sen, B. Zwiebach
"Ghost structure ..." hepth/0111129

D. J. Gross, W. Taylor
"Split string Field theory" I. II hepth/0105059
0106036

H. Hata, T. ~~Kawano~~ "Open string states ..." hepth/0108150

K. Okuyama "Ghost kinetic operator..." hepth/0201015
"Ratio of tensions ..." hepth/0201136

N. Moeller "Some exact results..." hepth/0110204

Y. Okawa "Open string states..." hepth/0204012

Moyal representation of SFT

I. Bars "Map of Witten's * to Moyal's *", hepth/0106157

M. Douglas, H. Liu, G. Moore, B. Zwiebach
"Open string star ..." hepth/0202087

B. Feng, Y.-H. He, N. Moeller hepth/0203175

B. Chen, F.-L. Lin hepth/0204233

D. M. Belov hepth/0204164

SFT with B field

E. Witten hepth/0006071

M. Schnabl hepth/0010034

F. Sugino hepth/9912254

T. Kawano, T. Takahashi hepth/9912274

L. B., D. Mamone, M. Salizzoni hepth/0201060
0203188
0207046

If we restrict Φ to

$$|\Phi\rangle = \int d^d k \left(\phi(k) + A_\mu(k)\, \alpha_{-1}^\mu \right) c_1 |k\rangle$$

the action becomes (Siegel gauge $b_0|\Phi\rangle = 0$)

$$S = \frac{1}{g_0^2} \int d^d x \Bigg(-\frac{1}{2} \partial_\mu \phi\, \partial^\mu \phi + \frac{1}{2\alpha'} \phi^2 - \frac{1}{2} \partial_\mu A_\nu\, \partial^\mu A^\nu$$

$$- \frac{1}{3} \left(\frac{3\sqrt{3}}{4} \right)^3 \tilde{\phi}^3(x) - \frac{3\sqrt{3}}{4}\, \tilde{\phi}\, \tilde{A}_\mu \tilde{A}^\mu +$$

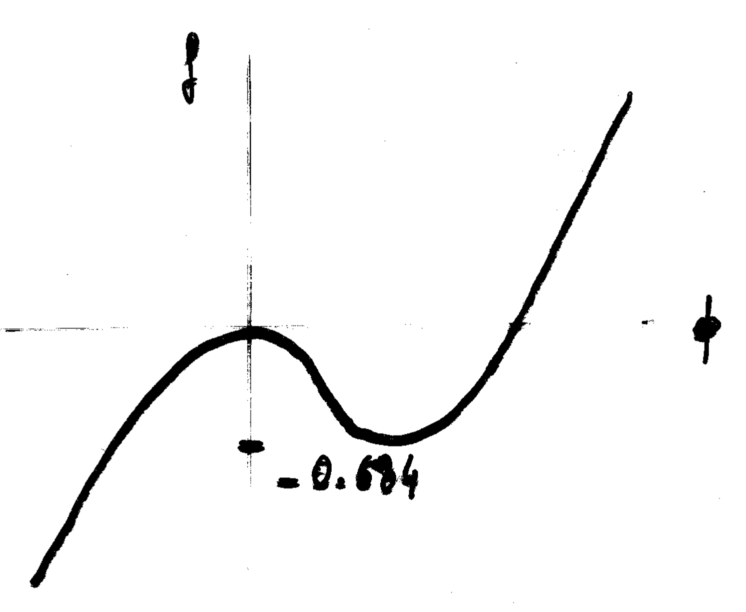
$$- \frac{3\sqrt{3}}{8} \alpha' \left(\partial_\mu \partial_\nu \tilde{\phi}\, \tilde{A}^\mu \tilde{A}^\nu + \tilde{\phi}\, \partial_\mu \tilde{A}^\nu \partial_\nu \tilde{A}^\mu - 2\, \partial_\mu \tilde{\phi}\, \partial_\nu \tilde{A}^\mu \tilde{A}^\nu \right) \Bigg)$$

where

$$\tilde{f}(x) = e^{-\alpha' \ln \frac{4}{3\sqrt{3}}\, \partial_\mu \partial^\mu} f(x)$$

Considering only the tachyon and dropping derivatives

$$S \longrightarrow \frac{1}{g_0^2} \int d^d x \left(\frac{1}{2\alpha'} \phi^2 - \frac{1}{3} \left(\frac{3\sqrt{3}}{4} \right)^3 \phi^3 \right) \equiv - \frac{f(\phi) V}{2\pi^2 \alpha'^3}$$



Vertex operators:

$$V_T = \int_{\partial\Sigma} dz\, e^{ik\cdot X} \qquad V_A = \int_{\partial\Sigma} \frac{dz}{z} A_\mu \partial X^\mu e^{ikX}$$

On-shell amplitudes

$$\langle V_1 \cdots V_N \rangle \qquad k_i^2 = -M_i^2$$

can be computed $\left(\Rightarrow \text{EOM} \Rightarrow \text{Action} \right.$ (in principle) $\left. \right)$

At one-loop we need off-shell information

$$\Downarrow$$

String Field Theory

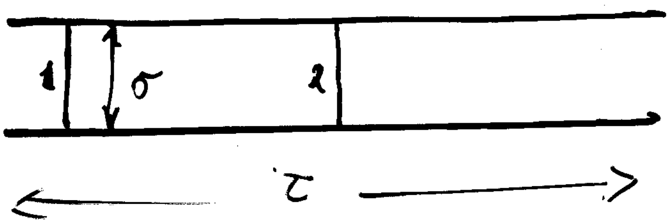
Bosonic Open String Theory

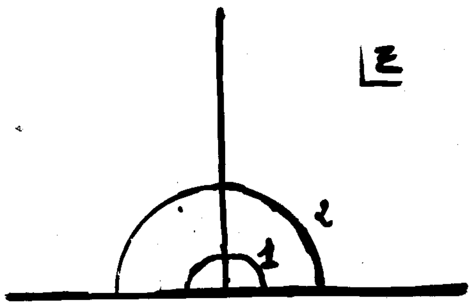
Action

$$S = \frac{1}{2\pi\alpha'} \int_\Sigma d^2z \, \partial X^\mu \bar{\partial} X_\mu$$

$$S_g = \frac{1}{2\pi} \int_\Sigma d^2z \, b \bar{\partial} c$$

World-sheet:





$0 \leq \sigma \leq \pi \qquad -\infty < \tau < +\infty$

$z = e^{t+i\sigma} \qquad t = i\tau$

Oscillator basis expansion:

$$X^\mu(z, \bar{z}) = x^\mu - i\alpha' p^\mu \ln|z|^2 + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} \left(z^{-n} + \bar{z}^{-n} \right)$$

Dirac brackets:

$$[\alpha_m^\mu, \alpha_n^\nu] = m \, \delta_{m+n,0} \, \eta^{\mu\nu}$$

$$[x^\mu, p^\nu] = i \eta^{\mu\nu}$$

$$\{b_m, c_n\} = \delta_{m+n,0}$$

Bosonic Open String Field Theory (D=26)

Action

$$S = -\frac{1}{g_0^2}\left(\frac{1}{2}\int \psi * Q_B \psi + \frac{1}{3}\int \psi * \psi * \psi\right)$$

where

$$Q_B^2 = 0$$

$$\int Q_B \psi = 0$$

$$(A * B) * C = A * (B * C)$$

$$Q_B(A * B) = (Q_B A) * B + (-1)^{|A|} A * (Q_B B)$$

Gauge invariance:

$$\delta\psi = Q_B \Lambda + \psi * \Lambda - \Lambda * \psi$$

By definition $|A| \equiv$ Grassmannality of A

$$\#_g(\psi) = \#_g(Q_B) = 1$$

$$\#_g(\Lambda) = 0$$

$$\#_g(*) = 0$$

$$\#_g\left(\int\right) = -3$$

Definitions:

1) The vacuum ($SL(2,\mathbb{R})$ invariant)

$$\alpha_n^\mu |0\rangle = 0 \qquad n \geq 0$$

$$c_n |0\rangle = 0 \qquad n > 1$$

$$b_n |0\rangle = 0 \qquad n \geq -1$$

2) The string Field

$$\Psi[x(\sigma)]$$

or

$$|\Psi\rangle = \left(\phi(x) + A_\mu(x)\,\alpha_{-1}^\mu + B_{\mu\nu}(x)\,\alpha_{-1}^\mu \alpha_{-1}^\nu + \dots\right) c_1 |0\rangle$$

Relation between the two: define

$$a_n^\mu = \frac{1}{\sqrt{n}}\alpha_n^\mu \qquad a_n^{\mu +} = \frac{1}{\sqrt{n}}\alpha_{-n}^\mu$$

$$\hat{x}_n = \frac{i}{\sqrt{2n}}\left(a_n - a_n^+\right) \qquad \hat{p}_n = \sqrt{\frac{n}{2}}\left(a_n + a_n^+\right)$$

$$\hat{x}(\sigma) = \hat{x}(\sigma, \tau = 0) = \hat{x}_0 + \sqrt{2}\sum_{n=1}^{\infty} \hat{x}_n \cos n\sigma$$

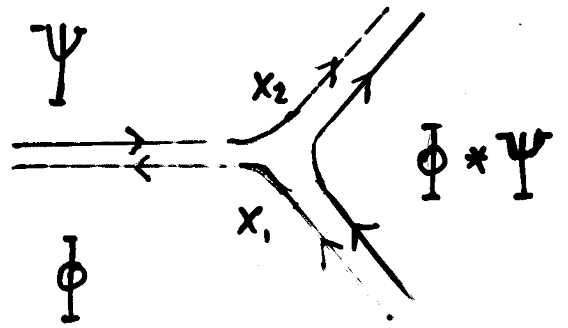
then

$$\Psi[x(\sigma)] = \langle \hat{x}(\sigma) | \Psi\rangle$$

$$|\hat{x}(\sigma)\rangle = \exp\sum_{n=1}\left(-\frac{1}{2}n\, x_n x_n - x_0^2 - i\sqrt{2n}\, a_n^+ x_n - 2i a_0^+ x_0 + \frac{1}{2} [illegible]\right)$$

3) The $*$ product.

Star product of $\Phi[x_1]$ with $\Psi[x_2]$ means identifying R half of x_1 with L half of x_2 and integrating over



- First formulation (functional)

$$(\Phi * \Psi)[z(\sigma)] = \int \Phi[x(\sigma)]\, \Psi[y(\sigma)] \prod_{\frac{\pi}{2}\le\sigma\le\pi} \delta[x(\sigma)-y(\pi-\sigma)] \prod_{\frac{\pi}{2}\le\sigma\le\pi} dx(\sigma) \prod_{0\le\sigma\le\frac{\pi}{2}} dy(\sigma)$$

$$z(\sigma) = x(\sigma) \qquad 0\le\sigma\le\frac{\pi}{2}$$

$$z(\sigma) = y(\sigma) \qquad \frac{\pi}{2}\le\sigma\le\pi$$

- Second formulation (operator)

3-string vertex $\langle V_3|$

$$\boxed{\begin{aligned}\langle V_3| = {}_1\langle 0|\, c_{-1}^{(1)} c_0^{(1)} \otimes {}_2\langle 0|\, c_{-1}^{(2)} c_0^{(2)} \otimes {}_3\langle 0|\, c_{-1}^{(3)} c_0^{(3)} \cdot \int dp_1\, dp_2\, dp_3\, \delta(p_1+p_2+p_3) \\ \cdot \exp -\left(\frac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n\ge 0} \eta_{\mu\nu}\, a_m^{(r)\mu} V_{mn}^{rs} a_n^{(s)\nu} + \sum_{r,s=1}^{3} \sum_{\substack{n\ge 1 \\ m\ge 0}} c_n^{(r)} \tilde{V}_{nm}^{rs} b_m^{(s)} \right)\end{aligned}}$$

Neumann coefficients

$$\left(\frac{1+ix}{1-ix}\right)^{1/3} = \sum_{n\ \text{even}} A_n x^n + i \sum_{n\ \text{odd}} A_n x^n$$

$$\left(\frac{1+ix}{1-ix}\right)^{2/3} = \sum_{n\ \text{even}} B_n x^n + i \sum_{n\ \text{odd}} B_n x^n$$

$$N^{r,\pm r}_{nm} = \begin{cases} \frac{1}{3(n\pm m)} (-1)^n \left(A_n B_m \pm B_n A_m\right) & m+n \text{ even} \quad n \neq m \\ 0 & m+n \text{ odd} \end{cases}$$

$$N^{r,\pm(r+1)}_{nm} = \begin{cases} \frac{1}{6(n\pm m)} (-1)^{n+1} \left(A_n B_m \pm B_n A_m\right) & m+n \text{ even} \quad n \neq m \\ \frac{1}{6(n\pm m)} \sqrt{3} \left(A_n B_m \mp B_n A_m\right) & m+n \text{ odd} \end{cases}$$

$$N^{r,\pm(r-1)}_{nm} = \begin{cases} \frac{1}{6(n\pm m)} (-1)^{n+1} \left(A_n B_m \mp B_n A_m\right) & m+n \text{ even} \quad m \neq n \\ -\frac{1}{6(n\mp m)} \sqrt{3} \left(A_n B_m \pm B_n A_m\right) & m+n \text{ odd} \end{cases}$$

$$V^{rr}_{nm} = -\sqrt{nm}\left(N^{rr}_{nm} + N^{r,-r}_{nm}\right) \qquad m \neq n, \quad m, n \neq 0$$

$$V^{rr}_{nn} = -\frac{1}{3}\left(2 \sum_{k=0}^{n} (-1)^{n-k} A_k^2 - (-1)^n - A_n^2\right) \qquad n \neq 0$$

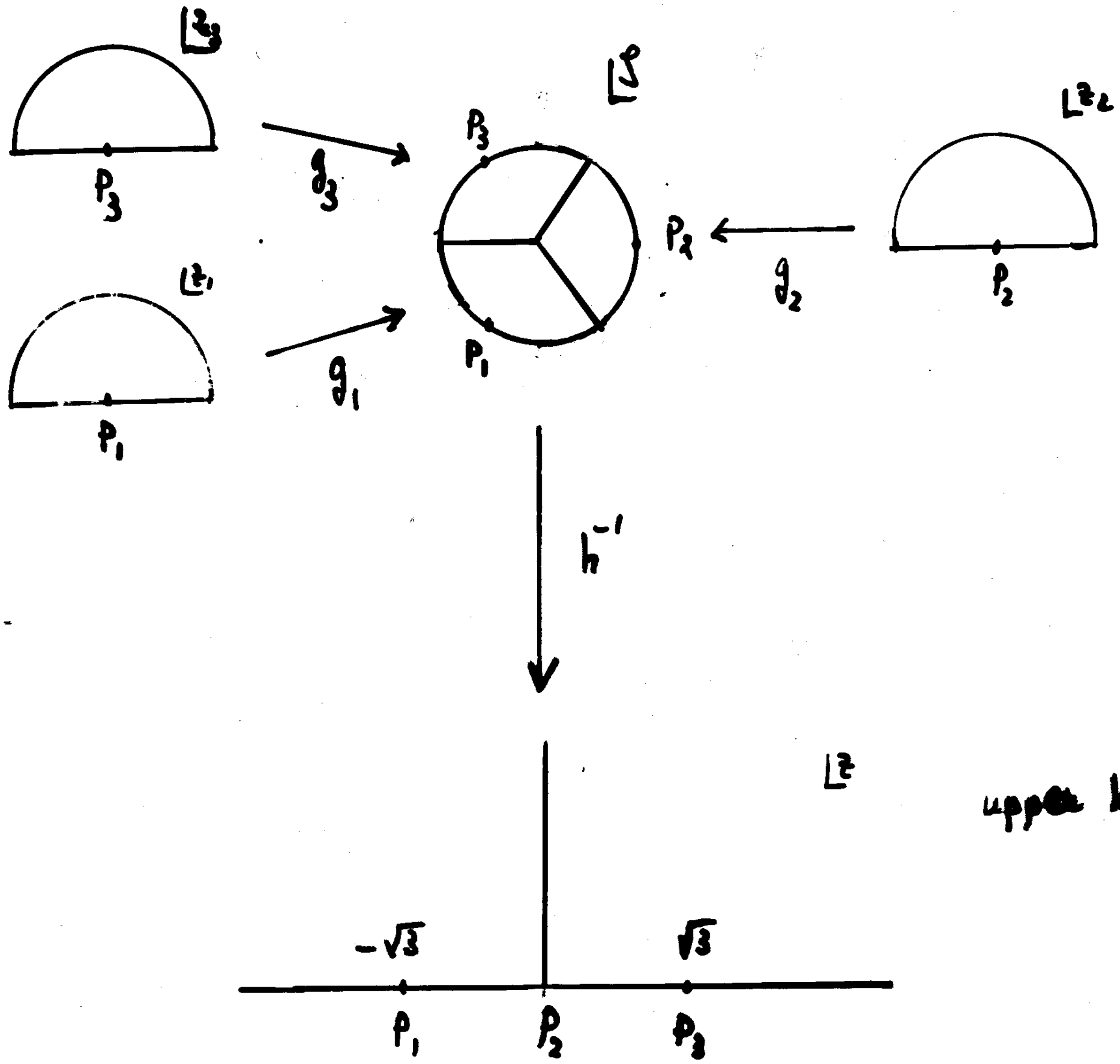
$$V^{r\,r+1}_{nn} = V^{r\,r+2}_{nn} = \frac{1}{2}\left((-1)^n - V^{rr}_{nn}\right) \qquad n \neq 0$$

$$V^{rr}_{0n} = -\sqrt{2n}\left(N^{rr}_{0n} + N^{r,-r}_{0n}\right) \qquad n \neq 0$$

$$V^{rr}_{00} = \ln \frac{27}{16}$$

- <u>Third formulation</u>

CFT formulation



upper half-plane H

$$g_k(z_k) = e^{\frac{2\pi i}{3}(k-2)} \left(\frac{1+i z_k}{1-i z_k}\right)^{2/3} \qquad z = h^{-1}(\xi) = -i\,\frac{\xi-1}{\xi+1}$$

$$f_k(z_k) = h^{-1} \circ g_k(z_k)$$

Then

$$\int \Phi * \Phi * \Phi = \langle f_1 \circ \Phi(0)\; f_2 \circ \Phi(0)\; f_3 \circ \Phi(0) \rangle_H$$

4) The BRST charge

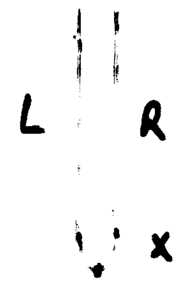
$$\boxed{Q_B = \sum_{n=-\infty}^{+\infty} c_n L^{(m)}_{-n} + \sum_{m,k} \frac{m-k}{2} :c_m c_k b_{-m-k}: - c_0}$$

$Q_B^2 = 0$ in $D=26$

$\{Q_B , b_0\} = L_0^{tot} \longrightarrow$ Siegel gauge $b_0|\psi\rangle = 0$

5) Integration

Integration corresponds to identifying L and R of string and integrating over



$\Leftrightarrow \quad \int \Phi[x] = \langle I | \Phi\rangle$

where

$$I[x(\sigma)] = \langle x(\sigma) | I \rangle = \prod_{0\leq\sigma\leq\pi/2} \delta\big(x(\sigma) - x(\pi-\sigma)\big)$$

More explicitly

$$\int \Phi \equiv \int \mathcal{D}x(\sigma) \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta\big(x(\sigma) - x(\pi-\sigma)\big)\, \Phi[x(\sigma)]$$

In operator language $\langle I | \equiv \langle I_m | \otimes \langle I_g |$;

$$\langle I_m | = \langle 0 | e^{-\frac{1}{2}\sum_n a_n C_{nm} a_m} \qquad C_{nm} = (-1)^n \delta_{nm}$$

$$\langle I_g | = \langle 0 | e^{-\sum_{n=1}^{\infty} (-1)^n c_n b_n}$$

where $\left[a_m^{(r)\mu}, a_n^{(s)\nu+}\right] = \eta^{\mu\nu}\delta_{mn}\delta^{rs}$.

$\hat{p}|p\rangle = p|p\rangle \quad , \quad \langle p|p'\rangle = \delta(p+p')$

Then

$$\boxed{{}_3\langle\phi * \psi| = \langle V_3|\phi\rangle_1|\psi\rangle_2}$$

where

$$\langle\phi| = bpz(|\phi\rangle)$$

Rules for bpz:

$bpz(\alpha_{-m}^{\mu}) = -(-1)^m \alpha_m^{\mu}$

$bpz(c_{-m}) = -(-1)^m c_m$

$bpz(b_{-m}) = (-1)^m b_m$

Use:

$$\langle 0| e^{\lambda_i a_i - \frac{1}{2} a_i P_{ij} a_j} \, e^{\mu_i a_i^+ - \frac{1}{2} a_i^+ Q_{ij} a_j^+} |0\rangle =$$

$$= (\det K)^{-1/2} \, e^{\mu^T K^{-1}\lambda - \frac{1}{2}\lambda^T Q K^{-1}\lambda - \frac{1}{2}\mu^T K^{-1} P \mu}$$

with $\quad K = 1 - PQ$

Level truncation

level	$f(T_0)$
(0,0)	−0.684
(2,4)	−0.949
(2,6)	−0.959
(4,8)	−0.986
(4,12)	−0.988
(6,12)	−0.99514
(6,18)	−0.99518
(8,16)	−0.99777
(8,20)	−0.99793
(10,20)	−0.99912

Some examples

- $|I\rangle$ is the identity for the $*$ product

$$\left(\Phi * I\right)[z(\sigma)] = \int \Phi[x(\sigma)] \prod_{0\le\sigma\le\frac{\pi}{2}} \delta\big(y(\sigma)-y(\pi-\sigma)\big) \cdot$$

$$\cdot \prod_{\frac{\pi}{2}\le\sigma\le\pi} \delta\left[x(\sigma)-y(\pi-\sigma)\right] \prod_{\frac{\pi}{2}\le\sigma\le\pi} dx(\sigma) \prod_{0\le\sigma\le\frac{\pi}{2}} dy(\sigma)$$

$$= \int \Phi[x(\sigma)] \prod_{\frac{\pi}{2}\le\sigma\le\pi} \delta\left[x(\sigma)-y(\sigma)\right] \prod_{\frac{\pi}{2}\le\sigma\le\pi} dx(\sigma)$$

$$= \Phi[y(\sigma)] \quad \tfrac{\pi}{2}\le\sigma\le\pi \qquad = \Phi[x(\sigma)] \quad 0\le\sigma\le\tfrac{\pi}{2}$$

$$= \Phi[z(\sigma)]$$

Another representation of $|I\rangle$:

$$|I\rangle = e^{L_{-2}-\frac{1}{2}L_{-4}+\frac{1}{2}L_{-6}-\frac{7}{12}L_{-8}\cdots}|0\rangle$$

SFT action and properties of Q_B

$$S(\Phi) = -\frac{1}{g_o^2}\left[\frac{1}{2}\langle \Phi, Q_B \Phi\rangle + \frac{1}{3}\langle \Phi, \Phi * \Phi\rangle\right.$$

BRST charge Q_B:

$$Q_B^2 = 0$$

$$Q_B(A*B) = (Q_B A)*B + (-1)^{|A|} A*(Q_B B)$$

$$\langle Q_B A, B\rangle = -(-1)^{|A|}\langle A, Q_B B\rangle$$

Inner product:

$$\langle A, B\rangle = (-1)^{|A||B|}\langle B, A\rangle$$

$$\langle A, B*C\rangle = \langle A*B, C\rangle$$

Associative $*$ product

$$A*(B*C) = (A*B)*C$$

$|A|$ is the Grassmannality of A

Vacuum String Field Theory

Defines a SFT corresponding to closed string vacuum. Just shift

$$\Phi = \Phi_0 + \tilde{\Phi} \qquad \Phi_0 \text{ corresponds to } T_0$$

Then

$$S(\Phi_0+\tilde{\Phi}) = -V_{25}T_{25} - \frac{1}{g_0^2}\int\Big[\frac{1}{2}(\Phi_0+\tilde{\Phi})*Q(\Phi_0+\tilde{\Phi}) + \\ +\frac{1}{3}(\Phi_0+\tilde{\Phi})*(\Phi_0+\tilde{\Phi})*(\Phi_0+\tilde{\Phi})\Big]$$

$$= -\frac{1}{g_0^2}\int\Big[\frac{1}{2}\tilde{\Phi}*\mathcal{Q}\tilde{\Phi} + \frac{1}{3}\tilde{\Phi}*\tilde{\Phi}*\tilde{\Phi}\Big]$$

where

$$\mathcal{Q}\tilde{\Phi} = Q_B\tilde{\Phi} + \frac{1}{2}\left(\Phi_0*\tilde{\Phi} + \tilde{\Phi}*\Phi_0\right)$$

Possible field redefinition

$$\tilde{\Phi} = e^K\Psi$$

Summing up <u>we postulate</u> at the closed string vacuum

$$S = -\frac{1}{g_0^2}\int\Big[\frac{1}{2}\Psi*\mathcal{Q}\Psi + \frac{1}{3}\Psi*\Psi*\Psi\Big]$$

The new BRST charge $\mathcal{Q}$ satisfies

$$\mathcal{Q}^2 = 0 \qquad \mathcal{Q}(\psi*\chi) = \mathcal{Q}\psi*\chi + (-1)^\psi\,\psi*(\mathcal{Q}\chi)$$

The new BRST charge must satisfy

$$\mathcal{Q}^2 = 0$$

$$\mathcal{Q}(A * B) = (\mathcal{Q}A) * B + (-1)^{|A|} A * (\mathcal{Q}B)$$

$$\langle \mathcal{Q}A, B \rangle = -(-1)^{|A|} A * (\mathcal{Q}B)$$

and

- $\mathcal{Q}$ must have <u>vanishing cohomology</u> (no open string states)

- $\mathcal{Q}$ must be <u>universal</u> (no dependence on BCFT)

Examples of $\mathcal{Q}$'s:

- $\mathcal{Q} = 0$

- $\mathcal{Q} \equiv \mathcal{C}_n = c_n + (-1)^n c_{-n}$ $\qquad n = 0, 1, 2, \ldots$

- $\mathcal{Q} \equiv \sum_{n=0} a_n \mathcal{C}_n$.

<u>Proof</u>: define $\mathcal{B}_n \equiv \frac{1}{2}\left(b_n + (-1)^n b_{-n}\right) \longrightarrow \{\mathcal{C}_n, \mathcal{B}_n\} = 1$

Therefore, if $\mathcal{C}_n \psi = 0 \longrightarrow \psi = \mathcal{C}_n(\mathcal{B}_n \psi) = \{\mathcal{C}_n, \mathcal{B}_n\} \psi$

Now search for classical solution of EOM of VSFT

$$\mathcal{Q}\Psi = -\Psi * \Psi$$

<u>Ansatz</u>

$$\Psi = \Psi_m * \Psi_g$$

So EOM splits

$$\mathcal{Q}\Psi_g = -\Psi_g *^g \Psi_g \qquad \Psi_m = \Psi_m *^m \Psi_m$$

and

$$S|_\Psi = -\frac{1}{6g_0^2} \langle \Psi_g | \mathcal{Q}\Psi_g \rangle_g \langle \Psi_m | \Psi_m \rangle_m \equiv K \langle \Psi_m | \Psi_m \rangle_m$$

Method of Kostelecky - Potting

Three string vertex $|V_3\rangle$:

$$|V_3\rangle = \int d^{26}p_{(1)}\, d^{26}p_{(2)}\, d^{26}p_{(3)}\, \delta^{(26)}\left(p_{(1)}+p_{(2)}+p_{(3)}\right) e^{-E} |0,p\rangle_{1,2,3}$$

with

$$E = \frac{1}{2}\sum_{\substack{r,s=1\\ m,n\geq 1}}^{3} \eta_{\mu\nu}\, a_m^{(r)\mu\dagger} V_{mn}^{rs} a_n^{(s)\nu\dagger} + \sum_{\substack{r,s=1\\ n\geq 1}}^{3} \eta_{\mu\nu}\, p_{(r)}^{\mu} V_{0n}^{rs} a_n^{(s)\nu\dagger} +$$

$$+ \frac{1}{2}\sum_{r=1}^{3} \eta_{\mu\nu}\, p_{(r)}^{\mu} V_{00}^{rr} p_{(r)}^{\nu} .$$

and

$$|0,p\rangle_{123} = |p_{(1)}\rangle \otimes |p_{(2)}\rangle \otimes |p_{(3)}\rangle$$

For space-time translational invariant solutions

$$E = \frac{1}{2}\sum_{\substack{r,s=1\\ m,n\geq 1}}^{3} \eta_{\mu\nu}\, a_m^{(r)\mu\dagger} V_{mn}^{rs} a_n^{(s)\nu\dagger}$$

<u>Ansatz</u>:

$$|\Psi_m\rangle = \mathcal{N}^{26}\, e^{-\frac{1}{2}\eta_{\mu\nu}\sum_{m,n\geq 1} S_{mn} a_m^{\mu\dagger} a_n^{\nu\dagger}} |0\rangle$$

Now impose

$$|\Psi_m *^m \Psi_m\rangle_3 \equiv {}_1\langle\Psi_m|\,{}_2\langle\Psi_m|V_3\rangle = |\Psi_m\rangle_3$$

Get equation

$$|\Psi_m * \Psi_m\rangle_3 = \mathcal{N}^{52} \det\left[(1-\Sigma\mathcal{V})^{-1/2}\right]^{26} \cdot$$

$$\cdot \exp\left[-\frac{1}{2}\eta_{\mu\nu}\left\{\chi^{\mu T}\frac{1}{1-\Sigma\mathcal{V}}\Sigma\chi^{\nu} + a^{(3)\mu\dagger}\cdot V^{33}\cdot a^{(3)\nu\dagger}\right\}\right]|0\rangle_3$$

where

$$\Sigma = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \qquad \mathcal{V} = \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{pmatrix}$$

$$\chi^{\mu T} = \left(a^{(3)\mu\dagger}V^{31},\ a^{(3)\mu\dagger}V^{32}\right) \qquad \chi^{\mu} = \begin{pmatrix} V^{13}a^{(3)\mu\dagger} \\ V^{23}a^{(3)\mu\dagger} \end{pmatrix}$$

Equating and using $V^{r+1,s+1} = V^{r,s}$ mod 3

$$(*) \qquad S = V^{11} + (V^{12}, V^{21})\frac{1}{1-\Sigma\mathcal{V}}\Sigma\begin{pmatrix} V^{21} \\ V^{12} \end{pmatrix}$$

Solve for S. seems hopeless

But . . . define

$$X^{rs} = C\, V^{rs}$$

$$C_{nm} = (-1)^n \delta_{nm}$$

$$\left[X^{rs}, X^{r's'}\right] = 0$$

Set

$$X = X^{11} \qquad T = CS$$

then $(*)$ becomes

$$(T-1)\left(X T^2 - (1+X)T + X\right) = 0$$

i.e.

$$S = CT \qquad T = \frac{1}{2X}\left(1 + X - \sqrt{(1+3X)(1-X)}\right)$$

Finally the solution is

$$\boxed{|\Psi_m\rangle = \left(\det(1-X)\,\det(1+T)\right)^{13} e^{-\frac{1}{2}\sum_{m,n\geq 1} a_m^\dagger S_{mn} a_n^\dagger}|0\rangle}$$

and

$$\boxed{S|_\Psi = K\, \frac{V^{(26)}}{(2\pi)^{26}}\left(\det(1-X)^{3/4}\,\det(1+3X)^{1/4}\right)^{-26}}$$

$|\Psi_m\rangle$ is identified with the D25 - brane.

The ghost sector (Hata, Kawano)

$$\mathcal{Q}\Psi_g = -\Psi_g *^g \Psi_g \qquad\qquad \mathcal{Q} = c_0 + \sum_{n=1}^{\infty} f_n \mathcal{C}_n$$

In Siegel gauge, $b_0|\Psi_g\rangle = 0$, we set $|\Psi_g\rangle = b_0|\phi_g\rangle$.
Then

$$|\phi_g\rangle_3 + {}_1\langle\phi_g|\,{}_2\langle\phi_g|\hat V_3\rangle = 0 \qquad \text{i.e.} \qquad |\phi_g\rangle + |\phi_g\rangle * |\phi_g\rangle = 0$$

$$\sum_{n\geq 1} f_n \mathcal{C}_n |\phi_g\rangle_3 + {}_1\langle\phi_g|\,{}_2\langle\phi_g|\sum_{r=1}^{3}\sum_{n\geq 1} c_n^{\dagger(r)}\, \tilde V_{n0}^{r3}|\hat V_3\rangle = 0$$

where

$$|V_3\rangle = e^{\sum_{n,m\geq 1} c_n^{(r)\dagger}\, \tilde V_{nm}^{rs}\, b_m^{(s)\dagger}}|0\rangle_1|0\rangle_2|0\rangle_3$$

Then

$$|\phi_g\rangle = \frac{1}{\det(1-\tilde S\tilde V)}\, e^{\sum_{n,m\geq 1} c_n^{\dagger}\, \tilde S_{nm}\, b_m^{\dagger}}|0\rangle$$

$$\underbrace{f}_{\{f_n\}} = \frac{1}{1-\tilde T}\left[\tilde v_0 + (\tilde X_{12}, \tilde X_{21})\,\frac{1}{1-\tilde T\tilde{\mathcal{X}}}\,\tilde T\begin{pmatrix}\tilde v_{21}\\ \tilde v_{12}\end{pmatrix}\right] \qquad (\tilde v_{rs})_n = \tilde V_{n0}^{rs}$$

$$\tilde{\mathcal{X}} = \begin{pmatrix}\tilde X & \tilde X_{12}\\ \tilde X_{21} & \tilde X\end{pmatrix}$$

One gets:

- $f_{2n+1} = 0 \qquad f_{2n} = 1 \qquad$ (Okuyama)

Fluctuation spectrum (Hata, Kawano)

Fluctuations are identified with solutions of linearized equations:

$$\phi = \phi_0 + \tilde{\phi} \qquad\qquad Q\tilde{\phi} + \phi_0 * \tilde{\phi} + \tilde{\phi} * \phi_0 = 0$$

Solution $|\tilde{\phi}\rangle = b_0|\phi_t\rangle$ with $p^2 = 1$ $(\alpha' = 1)$ $\longrightarrow$ tachyon

$$|\phi_t\rangle = \frac{\mathcal{N}_t}{\mathcal{N}_c}\, e^{-\sum_{n=1}^{\infty} t_n\, a_n^{\dagger} a_0} |\Xi\rangle .$$

with

$$t_{2n+1} = 0 \qquad\qquad t_{2n} = 1 \qquad\qquad (\text{i.e. } t_n = f_n)$$

Similarly massless vector mode

$$|\phi_v\rangle = \Big(\sum_{n\ \text{odd}} d_{n\mu}\, a_n^{\mu\dagger}\Big)|\phi_t\rangle$$

More accurate definition, see Okawa.

Lump solutions

They are supposed to represent D-(25-k)-branes.

k transverse directions, $\alpha = 1, \dots, k$.

Replace

$$|p\rangle = \frac{1}{\pi^{k/4}} \, e^{-\frac{1}{2} p^\alpha p^\alpha + \sqrt{2}\, a_0^{\alpha\dagger} p^\alpha - \frac{1}{2} a_0^{\alpha\dagger} a_0^{\alpha\dagger}} \, |\Omega\rangle$$

where

$$a_0^\alpha = \frac{1}{\sqrt{2}}\left(\hat{p}^\alpha - i \hat{x}^\alpha\right) \qquad a_0^{\alpha\dagger} = \frac{1}{\sqrt{2}}\left(\hat{p}^\alpha + i \hat{x}^\alpha\right)$$

$$[a_0^\alpha \,,\, a_0^{\beta\dagger}] = \delta^{\alpha\beta}$$

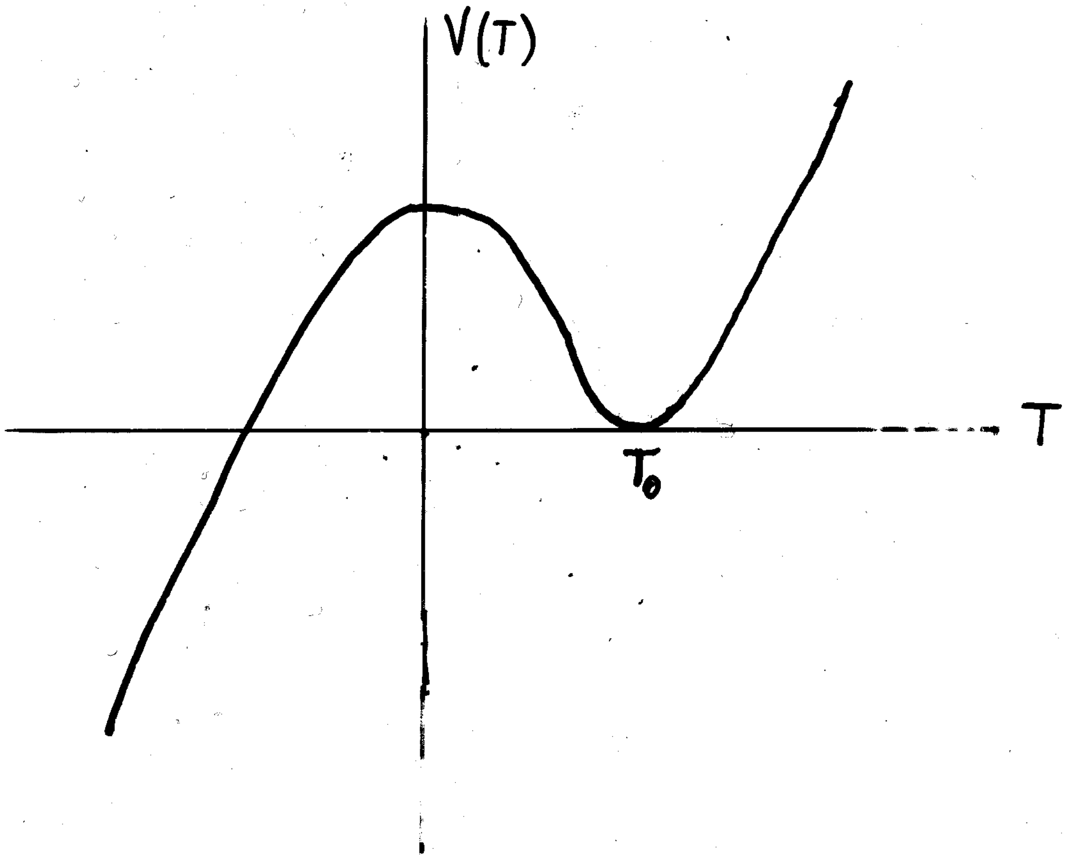
Integrate over p^α. The relevant vertex is:

$$|V_3\rangle = \exp\Big(-\frac{1}{2} \sum_{\substack{r,s\\ m,n\geq 1}} \eta_{\mu\nu}\, a_m^{(r)\mu\dagger} V_{mn}^{rs} a_n^{(s)\nu\dagger}\Big) |0,p\rangle_{123} \,\cdot$$

$$\cdot \left(\frac{\sqrt{3}}{(16\pi)} \cdots \left(V_{00}^{rr} + 1\right)\right)^{-k} \exp\Big(-\frac{1}{2} \sum_{\substack{r,s\\ M,N\geq 0}} a_M^{(r)\alpha\dagger} V'^{rs}_{MN} a_N^{(s)\alpha\dagger}\Big) |\Omega\rangle$$

$\mu = 0, \dots, 25-k-1$ $\qquad\qquad M \equiv \{0, m\}$

Sen's conjectures (on D=26 OBS)



$$V(T) = M\left(1 + f(T)\right) \qquad M = T_{25}$$

1) $f(T_0) = -1$

2) There exist soliton lumps that correspond to lower dimensional branes

3) The vacuum at T_0 is the closed string vacuum

The solution of $|\psi_m\rangle * |\psi_m\rangle = |\psi_m\rangle$ is

$$|\psi'_m\rangle = \left(\sqrt{\det(1-X)\det(1+T)}\right)^{26-k} e^{-\frac{1}{2}\sum_{m,n\geq 1} a_m^\dagger S_{mn} a_n^\dagger} |0\rangle$$
$$\otimes \left(\frac{\sqrt{3}}{(16\pi)^{1/4}}\left(V_{00}^{rr}+1\right)\right)^k \left(\det(1-X')\det(1+T')\right)^{k/2} e^{-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\prime\dagger} S_{MN} a_N^{\prime\dagger}} |0\rangle$$

Gives the action

$$S_{\psi'} = K \frac{V^{(26-k)}}{(2\pi)^{26-k}} \left(\det(1-X)^{3/4}\det(1+3X)^{1/4}\right)^{26-k} \cdot$$
$$\cdot \left(\frac{3}{(16\pi)^{1/2}}\left(V_{00}^{rr}+1\right)^2\right)^k \left(\det(1-X')^{3/4}\det(1+3X')^{1/4}\right)^k$$

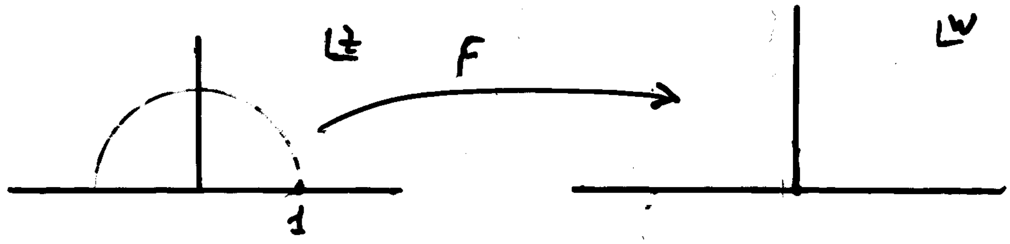
Ratio of tensions:

$$\frac{T_{24-k}}{2\pi\alpha' T_{25-k}} = \frac{3}{\sqrt{16\pi}}\left(V_{00}^{rr}+1\right)^2 \frac{\det(1-X')^{3/4}\det(1+3X')^{1/4}}{\det(1-X)^{3/4}\det(1+3X)^{1/4}}$$

Numerically this $= 1$. (Okuyama)

- **Surface states**

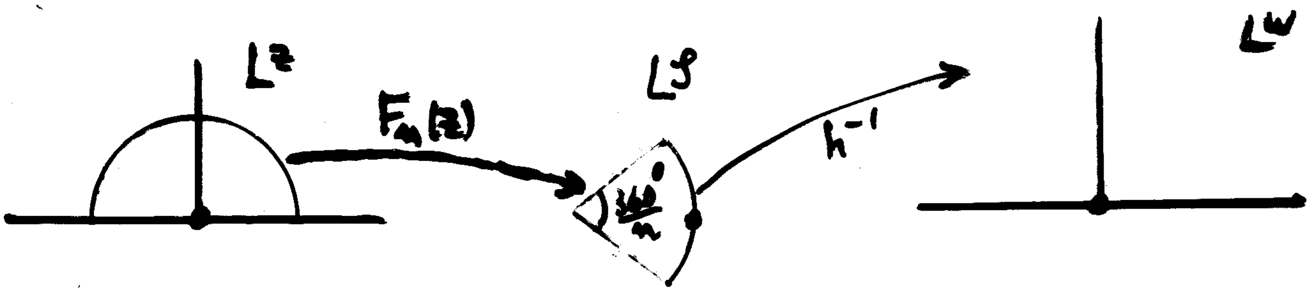
defined via conformal map $F(z)$ of the upper half disk to the upper half plane



$\langle f|$ is defined via

$$\langle f|\phi\rangle = \langle f\circ\phi(0)\rangle \qquad |\phi\rangle = \phi(0)|0\rangle$$

- **Wedge states**



$$F_n(z) = \left(\frac{1+iz}{1-iz}\right)^{2/n} \qquad h^{-1}(\xi) = -i\,\frac{\xi-1}{\xi+1}$$

$$f_n = h^{-1}\circ F_n(z) = \tan\left(\frac{2}{n}\arctan(z)\right)$$

Then

$$|n\rangle * |m\rangle = |n+m-1\rangle$$

and

$$|n=1\rangle = |I\rangle \qquad |n=\infty\rangle = \text{sliver}$$

Representation of wedge states $|n\rangle$

1) $\langle n|\phi\rangle \equiv \langle F_n \circ \phi(0)\rangle$ for any state $|\phi\rangle = \phi(0)|0\rangle$

$$F_n(z) = \frac{n}{2} \operatorname{tg}\left(\frac{2}{n} \operatorname{tg}^{-1}(z)\right)$$

2) $$|n\rangle = \exp\left(-\frac{n^2-4}{3n^2} L_{-2} + \frac{n^4-16}{30n^4} L_{-4} - \frac{(n^2-4)(176+128n^2+11n^4)}{1890 n^6} L_{-6} + \dots\right)|0\rangle$$

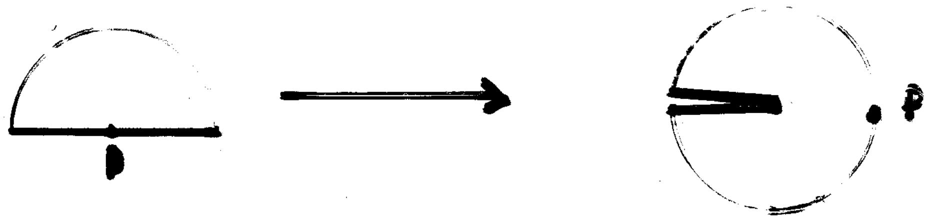
Star product of wedge states

$$|n\rangle * |m\rangle = |n+m-1\rangle$$

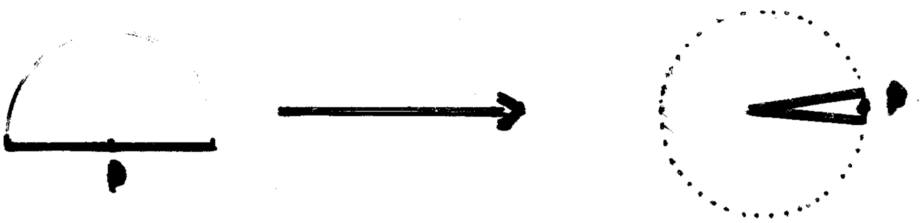
Two states satisfy $\psi * \psi = \psi$

$n=1$ <u>identity state</u> $|I\rangle \equiv |1\rangle$

$n=\infty$ <u>sliver state</u> $|\Xi\rangle \equiv |\infty\rangle$



$|I\rangle$



$|\infty\rangle$

Using the representation :

$$|\Xi\rangle = e^{\left(-\frac{1}{3}L_{-2}+\frac{1}{30}L_{-4}-\frac{11}{1890}L_{-6}+\frac{34}{467775}L_{-8}+\dots\right)}|0\rangle$$

and
$$L_{-n} = L^{m}_{-n} + L^{g}_{-n}$$

one gets

$$|\Xi\rangle = |\Xi_g\rangle \otimes |\Xi_m\rangle$$

$$|\Xi_m\rangle = \bar{\mathcal{N}}^{26}\exp\left(-\frac{1}{3}L^{m}_{-2}+\frac{1}{30}L^{m}_{-4}-\frac{11}{1890}L^{m}_{-6}+\dots\right)|0\rangle$$

Then

$$|\Xi_m\rangle *^{m} |\Xi_m\rangle = K\bar{\mathcal{N}}^{52}|\Xi_m\rangle$$

Now, choose $\bar{\mathcal{N}}$ so that

$$K\bar{\mathcal{N}}^{52} = 1$$

and compare

$$|\Psi_m\rangle = \mathcal{N}^{26}\, e^{-\frac{1}{2}\eta_{\mu\nu}\, a^{\mu\dagger}\cdot S\cdot a^{\nu\dagger}}|0\rangle$$

with

$$|\Xi_m\rangle = \bar{\mathcal{N}}^{26}\, e^{-\frac{1}{2}\eta_{\mu\nu}\, a^{\mu\dagger}\cdot \bar{S}\cdot a^{\nu\dagger}}|0\rangle$$

Numerically $\quad S_{mn} \simeq \bar{S}_{mn}$

Split String Field Theory

Treat separately the L and R half of the string. Define

$$\ell(\sigma) = x(\sigma) \qquad r(\sigma) = x(\pi-\sigma) \qquad 0 \le \sigma \le \frac{\pi}{2}$$

Neumann b.c. at $\sigma = \frac{\pi}{2}$ Dirichlet b.c at $\sigma = 0, \pi$

Then

$$\begin{cases} \ell(\sigma) = \sqrt{2} \sum_{n=0}^{\infty} \ell_{2n+1} \cos(2n+1)\sigma \\ r(\sigma) = \sqrt{2} \sum_{n=0}^{\infty} r_{2n+1} \cos(2n+1)\sigma \end{cases}$$

and

$$\begin{cases} x_{2n+1} = \frac{1}{2}\left(\ell_{2n+1} - r_{2n+1}\right) \\ x_{2n} = \frac{1}{2} \sum_{k=0}^{\infty} X_{2n,2k+1}\left(\ell_{2k+1} + r_{2k+1}\right) \end{cases}$$

$$\begin{cases} \ell_{2k+1} = x_{2k+1} + \sum_{n=0}^{\infty} X_{2k+1,2n}\, x_{2n} \\ r_{2k+1} = -x_{2k+1} + \sum_{n=0}^{\infty} X_{2k+1,2n}\, x_{2n} \end{cases}$$

In this we can define for any $\psi[x(\sigma)]$ an operator $\hat{\Psi}$

$$\Psi[x(\sigma)] \longrightarrow \hat{\Psi} = \int d\ell\, dr\, |\ell\rangle\, \psi[\ell, r]\, \langle r|$$

also

$$\langle x(\sigma)|\Psi\rangle = \langle \ell|\hat{\Psi}|r\rangle \qquad |\ell\rangle \equiv |\{\ell_{2n+1}\}\rangle$$

In particular

$$\int \Psi \longrightarrow \mathrm{Tr}(\hat{\Psi}) \qquad \Psi_1 * \Psi_2 \longrightarrow \hat{\Psi}_1 \hat{\Psi}_2$$

In the half-string formalism the sliver factorizes

$$\langle \vec{x} | = K_0^{26} \langle 0 | \, e^{-x\cdot E^{-2}\cdot x + 2i a\cdot E^{-1}\cdot x + \frac{1}{2} a\cdot a}$$

with

$$\hat{x}_n^\mu = \frac{i}{\sqrt{2n}} \left(a_n^\mu - a_n^{\mu +} \right), \qquad \hat{x} = \frac{i}{2} E\cdot (a - a^+) \qquad E_{nm} = \sqrt{\frac{2}{n}}\,\delta_{nm}$$

Then

$$\langle \vec{x} | \Xi \rangle = \tilde{\mathcal{N}}^{26} \, e^{-\frac{1}{2} x\cdot V\cdot x}$$

where

$$V = 2\, E^{-1} \frac{1-S}{1+S} E^{-1}$$

After passing to the half-string basis $x \to (x_L, x_R)$

$$\langle \vec{x} | \Xi \rangle = \tilde{\mathcal{N}}^{26} \, e^{-\frac{1}{2} x^L\cdot K\cdot x^L} \, e^{-\frac{1}{2} x^R\cdot K\cdot x^R}$$

with

$$K = A_+^T V A_+ = A_-^T V A_-$$

and

$$x_n^\mu = A^+_{nm}\, x_m^{L\mu} + A^-_{nm}\, x_m^{R\mu} \qquad n, m \geq 1$$

Noncommutative Solitons

Generic (ordinary) scalar field theory in D dim

$$S = \frac{1}{g^2}\int d^D x \left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right)$$

look for a classical solution ϕ_0 which is an extremum of the energy $E[\phi]$:

$$E(\lambda) = \frac{1}{g^2}\int d^D x \left(\frac{1}{2}\left(\partial\phi_0(\lambda x)\right)^2 + V\left(\phi_0(\lambda x)\right)\right) =$$

$$= \frac{1}{g^2}\int d^D x \left(\frac{1}{2}\lambda^{2-D}\left(\partial\phi_0(x)\right)^2 + \lambda^{-D}V\left(\phi_0(x)\right)\right)$$

Then

$$0 = \left.\frac{\partial E(\lambda)}{\partial\lambda}\right|_{\lambda=1} = -\frac{1}{g^2}\int d^D x \left(\frac{1}{2}(D-2)\left(\partial\phi_0\right)^2 + D\,V(\phi_0)\right)$$

For $D \geq 2$ and $V \geq 0$ this can only vanish if the kinetic and potential terms vanish separately.

On the contrary, <u>solitonic solutions exist in the corresponding noncommutative theories</u>

Moyal product in $\mathbb{R}^d$ $\qquad \theta^{\mu\nu} = -\theta^{\nu\mu}$

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)\cdot g(y)\Big|_{y=x}$$

In particular

$$x^\mu * x^\nu - x^\nu * x^\mu = i\,\theta^{\mu\nu}$$

$$e^{ipx} * e^{iqx} = e^{i(p+q)x}\, e^{-\frac{i}{2}p_\mu\theta^{\mu\nu}q_\nu}$$

Moyal product defines a n.c. associative algebra $\mathcal{A}_\theta$.

$$\int d^dx\, f*g = \int d^dx\, fg$$

GFT in n.c. $\mathbb{R}^d$

$$\delta_\lambda A_\mu = \partial_\mu\lambda + i\lambda * A_\mu - i A_\mu * \lambda$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i A_\mu * A_\nu - i A_\nu * A_\mu$$

Action

$$S = -\frac{1}{4g^2}\int d^dx\, \mathrm{Tr}(F*F)$$

if $\quad A_\mu = A_\mu^a t^a \quad , \quad (t^a)^\dagger = t^a$

Simple example in 2+1 D and $\theta \to \infty$.
Coordinates : x^1, x^2, t with $z = x^1 + i x^2$
Rescale $x^i \to x^i \sqrt{\theta}$, then

$$E = \frac{1}{g^2} \int d^2 z \left(\frac{1}{2} (\partial \phi)^2 + \theta V_*(\phi) \right)$$

In the limit $\theta \to \infty$

$$E = \frac{\theta}{g^2} \int d^2 z \, V(\phi)$$

Extremum

$$\frac{\partial V}{\partial \phi} = 0$$

Example (cubic potential):

$$m^2 \phi + b_3 \phi * \phi = 0$$

i.e.

$$\boxed{\phi_0 * \phi_0 = \phi_0}$$

Solution

$$\phi_0(x) = 2 e^{-r^2}, \qquad r^2 = x_1^2 + x_2^2$$

Rescaling back

$$\phi_0(x) = 2 e^{-\frac{r^2}{\theta}}$$

Noncommutative Solitons

Two noncommutative coordinates

$$[x^1, x^2]_* = i\theta$$

can be mimicked by two quantum operators $\hat{p}$, $\hat{q}$:

$$[\hat{q}, \hat{p}] = i$$

Then use Weyl quantization:

there is a 1-1 correspondence between the algebra of function with $*$ product and the algebra of operators in Hilbert space

Correspondence: $p, q \longleftrightarrow \hat{p}, \hat{q}$

For any classical function $f(p,q)$ introduce the Fourier transform

$$\hat{f}(k_q, k_p) = \int dp\, dq\; e^{i(k_q q + k_p p)}\, f(p,q)$$

and operator

$$U(k_q, k_p) = e^{-i(k_q \hat{q} + k_p \hat{p})}$$

The correspondence between classical functions $f(p,q)$ and quantum operators is given by:

$$f(q,p) \longleftrightarrow \hat{O}_f$$

$$\hat{O}_f(\hat{q},\hat{p}) = \frac{1}{(2\pi)^2}\int dk_p\, dk_q \; U(k_q,k_p)\, \tilde{f}(k_q,k_p)$$

$$f(q,p) = \int dk_p \; e^{-ip\,k_p} \; \langle q+\tfrac{k_p}{2}|\, \hat{O}_f(\hat{q},\hat{p})\,|q-\tfrac{k_p}{2}\rangle$$

Examples:

$$\int dq\, dp\, f(q,p) = 2\pi\, \mathrm{Tr}_{\mathcal{H}}\, \hat{O}_f = 2\pi \int dq\, \langle q|\, \hat{O}_f\, |q\rangle$$

$$\boxed{\hat{O}_f\, \hat{O}_g = \frac{1}{(2\pi)^2}\int dk_p\, dk_q\; U(k_q,k_p)\; \widetilde{f*g}\,(k_q,k_p) = \hat{O}_{f*g}}$$

$$[\hat{O}_f, \hat{O}_g] = \hat{O}_{f*g-g*f}$$

Consider the previous example $(\theta \to \infty)$

$$S = \int dt\, dx^1 dx^2 \;\; V_*(\phi)$$

Now, Weyl transform:

$$\phi \to \hat{O}_\phi \equiv \hat{\phi} \qquad\qquad S = 2\pi\theta \int dt\, \mathrm{Tr}_{\mathcal{H}}\, V(\hat{\phi})$$

The eq. of motion is: $V'(\phi) = 0$

$$V'(\phi) = \text{const}\;\; \phi(\phi - \lambda_1) \cdots (\phi - \lambda_{n-1}) = 0$$

Now, if $\hat{P}$ is a projector, the configuration

$$\hat{\phi} = \lambda_i \hat{P} \qquad\qquad \hat{P}^2 = \hat{P}$$

is a solution, since $\qquad E = 2\pi\theta\, V(\lambda_i)\, \mathrm{Tr}_{\mathcal{H}} \hat{P}$

$$\hat{P}(1 - \hat{P}) = 0$$

In general

$$\hat{\phi} = \Sigma_i \lambda_i \hat{P}_i \qquad\qquad \hat{P}_i \perp \hat{P}_j$$

is a non-trivial solution.

Let $a = \frac{\hat{q} + i\hat{p}}{\sqrt{2}}$, $[a, a^+] = 1$ and let

$|n\rangle$ be a basis of harmonic oscillator eigenstates.

Consider the operator $|n\rangle\langle m|$ and its Weyl transform

$$f_{n,m}(q,p) = \int dy \, e^{-ipy} \langle q + \tfrac{y}{2} | n\rangle \langle m | q - \tfrac{y}{2}\rangle$$

Adapting to $(q,p) = (x^1, x^2)$ one finds

$$f_{n,m}(r,\phi) = 2 e^{-r^2} \sqrt{\frac{n!}{m!}} (-1)^n (2r^2)^{\frac{m-n}{2}} e^{i\phi(m-n)} L_n^{m-n}(2r^2)$$

In particular

$$f_{0,0}(x_1, x_2) = 2 e^{-(x_1^2 + x_2^2)}$$

This corresponds to the projector

$$\hat{P} = |0\rangle\langle 0|$$

The energy of the corresponding solution is:

$$E = 2\pi\theta \, \mathrm{Tr}_{\mathcal{H}} V(\lambda_i \hat{P}) = 2\pi\theta \, V(\lambda_i) \Leftarrow \mathrm{Tr}_{\mathcal{H}}(\hat{P}) = 1$$

For generic n we have

$$P_n = |n\rangle\langle n| \longleftrightarrow \psi_n = f_{n,n}(r,\phi) = (-1)^n 2 L_n\left(\frac{r^2}{\theta}\right) e^{-\frac{r^2}{\theta}}$$

after rescaling back

$$x, y \longrightarrow \frac{x}{\sqrt{\theta}}, \frac{y}{\sqrt{\theta}} \qquad r = \sqrt{x^2+y^2}$$

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!} (-x)^k \qquad \text{Laguerre polyn.}$$

Many soliton solutions ...

$$E = 2\pi\theta\, V(\lambda_i)\, \mathrm{Tr}_{\mathcal{H}}\, \hat{P} \qquad \text{for} \quad \hat{\phi} = \lambda_i \hat{P}$$

Generalizations

- to more than 2+1 D:

$$\theta^{ij} = \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & 0 & \theta_2 & \\ & & -\theta_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

Regroup coordinates 2x2.

- solution generating tecnique

$$\mathcal{L} = 2\pi\theta\, \mathrm{Tr}_{\mathcal{H}}\, V(\phi)$$

is invariant under

$$\phi \to U\phi U^{+} \qquad UU^{+} = U^{+}U = 1, \quad U \in \mathcal{U}(\mathcal{H})$$

and maps solutions to solutions, since

$$V'(\phi) \to U\, V'(\phi)\, U^{+}$$

However, suppose that

$$U^\dagger U = 1 \qquad \text{but} \qquad U U^\dagger \neq 1 \quad \text{(non unitary isometry)}$$

Then (if no linear term in $V(\phi)$) still

$$\frac{dV}{d\phi} = 0 \implies U \frac{dV}{d\phi} U^\dagger = 0$$

under

$$\phi \longrightarrow U \phi U^\dagger$$

i.e. if ϕ_0 is a solution, then $U\phi_0 U^\dagger$ is a sol.

Example of <u>non-unitary isometry</u>: shift op.

$$S = \sum_{n=0}^{\infty} |n+1\rangle\langle n| \qquad\qquad S : |k\rangle \longrightarrow |k+1\rangle$$

Start from trivial solution $\phi_0 = \lambda_i I$

and apply $U = S^n$

$$\phi_0 \to U \phi_0 U^\dagger = S^n \lambda_i I \, S^{n\dagger} = \lambda_i (1 - P_n)$$

where

$$P_n = \sum_{k=0}^{n-1} |k\rangle\langle k|$$

This generates a new solution.

• Inclusion of derivatives and gauge fields.

Weyl transform of derivatives:

$$\hat{O}_{\partial_q f} = i\,[\hat{p}, \hat{O}_f]$$

$$\hat{O}_{\partial_p f} = -i\,[\hat{q}, \hat{O}_f]\,.$$

Since $[x^i, x^j]_\star = i\theta^{ij}$ we have

$$\partial_i \longrightarrow -i\,\theta_{ij}\,[\hat{x}^j,\ ] = -i\,\theta_{ij}\,\mathrm{ad}_{\hat{x}^j}$$

with $\theta_{ij}\theta^{jk} = \delta_i^{\ k}$.

This definition satisfies:

1) $[\partial_i, \partial_j] = 0$

2) $\partial_i x^j = \delta_i^{\ j}$

3) linearity

4) Leibnitz

Useful to introduce $z = \frac{1}{\sqrt{2}}(x^1 + i x^2)$, $a \equiv \frac{z}{\sqrt{\theta}}$

$$\partial \equiv \partial_z = -\,\theta^{-1/2}\,\mathrm{ad}_{a^+} \qquad \bar{\partial} \equiv \partial_{\bar{z}} = \theta^{-1/2}\,\mathrm{ad}_a$$

Similarly in $2n$ dimensions introduce z_α and

$$\partial_\alpha \equiv -\,\theta_\alpha^{-1/2}\,\mathrm{ad}_{a_\alpha}\,. \qquad \alpha = 1, 2, \ldots, n$$

Now we introduce a gauge Field A_i

$$[A_i, \phi]_* \longrightarrow [\hat{A}_i, \hat{\phi}]$$

Therefore covariant derivatives become

$$D_\alpha \phi \longrightarrow -\theta_\alpha^{-1/2}[\hat{a}_\alpha^+ + i\sqrt{\theta_\alpha}\hat{A}_\alpha, \hat{\phi}]$$

Set $\hat{C}_\alpha = \hat{a}_\alpha^+ + i\sqrt{\theta_\alpha}\hat{A}_\alpha$, then

$$D_\alpha \phi \longrightarrow -\theta_\alpha^{-1/2}[\hat{C}_\alpha, \hat{\phi}]$$

gauge transformation

$$\hat{C}_\alpha \longrightarrow U C_\alpha U^+ \qquad U \in \mathcal{U}(\mathcal{H})$$

Curvature:

$$F_\alpha \longrightarrow \theta_\alpha^{-1}\left([\hat{C}_\alpha, \hat{C}_\alpha^+] + 1\right)$$

Action

$$S = 2\pi \,\mathrm{Pf}(\theta^{ij}) \int dt \, \mathrm{Tr}_{\mathcal{H}}\left(-\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \frac{1}{2} D^\nu\hat{\phi}\, D_\nu\hat{\phi} - V(\hat{\phi})\right)$$

Generating soliton solutions starting from

$$\hat{\phi} = \phi_0 I \quad (\text{vacuum}) \qquad \hat{C} = \hat{a}^{+} \longrightarrow A = 0$$

and applying an "almost gauge" transformation

$$U^{+}U = 1 \qquad UU^{+} \neq 1 \qquad U = S^{n}$$

i.e.

$$\hat{\phi} = \phi_0 (1 - P_n) \qquad \hat{C} = S^{n} \hat{a}^{+} S^{n+} \qquad \hat{C}^{+} = S^{n} \hat{a} S^{n+}$$

Then, for instance,

$$\hat{F} = \frac{1}{\theta}\left([\hat{C}, \hat{C}^{+}] + 1\right) = \frac{1}{\theta}\left(S^{n}[\hat{a}^{+}, \hat{a}]S^{n+} + 1\right) = \frac{\hat{P}_n}{\theta}$$

and the energy is

$$E = 2\pi\theta \; \mathrm{Tr}\left(\frac{1}{2}\hat{F}^2 + \frac{1}{\theta}[\hat{C}, \hat{\phi}][\hat{C}^{+}, \hat{\phi}] + V(\hat{\phi} - \phi_0)\right)$$

$$= 2\pi\theta \, n \left(\frac{1}{2\theta^2} + V(-\phi_0)\right)$$

- Valid for any θ !!
- Energy $\to \infty$ as $\theta \to 0$

Bosonic D-branes as nc solitons

Adding a gauge Field to the purely tachyonic action, one ends up with:

higher derivatives

$$S = T_{25}\int d^{26}x\left[-V(\phi-1)\sqrt{-\det(g+2\pi\alpha' F)}+\sqrt{g}\,f(\phi-1)\,\partial^\mu\phi\,\partial_\mu\phi+\cdots\right]$$

$V(\phi-1)$ has a local max. at $\phi=0$ and $V(-1)=1$ and a local min. at $\phi=1$ with $V(0)=0$.

Now, let us switch on a B field in the $24, 25$-th direction.

$$B_{24,25}=b$$

So

$$g_{\mu\nu}\to G_{\mu\nu}$$
$$g_s\to G_s$$

and going to the operator action

$$S = 2\pi\theta\, T_{25}\frac{g_s}{G_s}\int d^{24}x\, \mathcal{L}_{nc}$$

$$\mathcal{L}_{nc} = \mathrm{Tr}_{\mathcal{H}}\left[-V(\phi-1)\sqrt{\det\left(G_{\mu\nu}+2\pi\alpha'(F+\Phi)_{\mu\nu}\right)}+\frac{1}{2}\sqrt{G}\,f(\phi-1)\,D^\mu\phi\, D_\mu\phi+\cdots\right]$$

We start from the vacuum solution

$$\phi = 1\,, \quad C = a^{+}\,, \quad A_\mu = 0 \qquad \mu = 0, 1, \ldots, 23$$

and apply the solution generating technique:

$$\phi \to U \phi U^{+}$$
$$C \to U C U^{+} \qquad\qquad F_{24,25} + \Phi_{24,25} = -\frac{[C, C^{+}]}{\theta}$$
$$A_\mu \to U A_\mu U^{+}$$

with

$$U^{+} U = 1 \qquad\qquad U U^{+} = 1 - P_n$$

Let us choose $U = S^n$. Then $\qquad \left(S = \sum_{k=0}^{\infty} |k+1\rangle\langle k| \right)$

$$\phi = S^n S^{+n} = 1 - P_n$$
$$C = S^n a^{+} S^{+n}$$
$$A_\mu = 0$$

From this follows

$$D\phi = 0 = \bar{D}\phi \qquad\qquad D\phi = -\frac{1}{\sqrt{\theta}}[C, \phi]$$

$$DF = 0 = \bar{D}F \qquad\qquad F = \frac{1}{\theta}\left([C, \bar{C}] + 1\right)$$

Therefore

$$V(\phi-1)\,[C, C^{+}]^{m} = V(-P_n)\,(1-P_n)^{m} \doteq V(-1)\,P_n\,(1-P_n) = 0$$

So only the 0-th order term contributes

$$S = (2\pi)^2 \alpha' \; T_{25} \int d^{24}x \; \mathrm{Tr}_{\mathcal{H}} \, P_n \equiv T \; V_{24}$$

i.e.

$$T = (2\pi)^2 \alpha' \, n \, T_{25} = n \, T_{23}^{0}$$

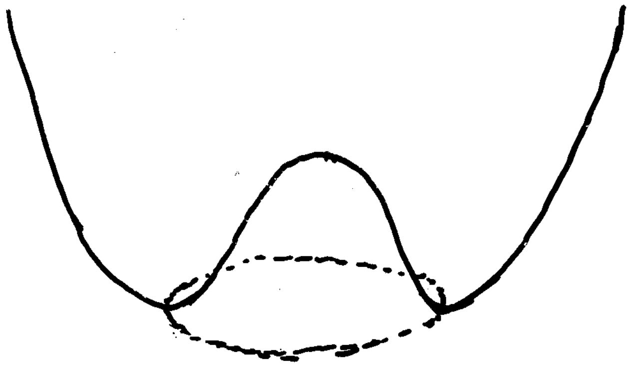
- Valid for any B
- There is a $U(n)$ symmetry subgroup, so there are $U(n)$ gauge field excitations.

The same construction works for type II branes.

Ex.: $D9-\overline{D9}$ system in IIB

The tachyon is complex: ϕ

The potential $V(\phi\phi^*-1)+V(\phi^*\phi-1)$ has a ring of minima.



There are two gauge fields: A^+, A^-.

The solution:

$$\phi = S^m S^{m+}$$

$$C^- = S^m a^+ S^{+m}$$

$$C^+ = S^m a^+ S^{+m}$$

$$A^+_\mu = A^-_\mu = 0$$

represents m D7-branes coincident with m $\overline{D7}$-branes

STAR ALGEBRA SPECTROSCOPY

PROBLEM: Diagonalize $X, X^{12}, X^{21}; T$

Use $K_1 = L_1 + L_{-1} \longrightarrow \mathcal{K}_1 = -(1+z^2)\frac{d}{dz}$

with properties

$$[K_1, X] = [K_1, X^{12}] = [K_1, X^{21}] = [K_1, T] = 0$$

Result:

$$K_1 v^{(\kappa)} = \kappa v^{(\kappa)} \qquad -\infty < \kappa < +\infty$$

$$v^{(\kappa)} = (v_1^{(\kappa)}, v_2^{(\kappa)}, \ldots)$$

with

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} v_n^{(\kappa)} z^n = \frac{1}{\kappa}\left(1 - e^{-\kappa \tan^{-1} z}\right)$$

Then

$$X v^{(\kappa)} = \mu(\kappa) v^{(\kappa)}, \qquad \mu(\kappa) = -\frac{1}{1 + 2\cosh\frac{\pi\kappa}{2}}$$

$$X^{12} v^{(\kappa)} = \mu^{12}(\kappa) v^{(\kappa)}, \qquad \mu^{12}(\kappa) = -\left(1 + e^{\frac{\pi\kappa}{2}}\right)\mu(\kappa)$$

$$X^{21} v^{(\kappa)} = \mu^{21}(\kappa) v^{(\kappa)}, \qquad \mu^{21}(\kappa) = -\left(1 + e^{-\frac{\pi\kappa}{2}}\right)\mu(\kappa)$$

$$T v^{(\kappa)} = \tau(\kappa) v^{(\kappa)}, \qquad \tau(\kappa) = -e^{-\frac{\pi|\kappa|}{2}}$$

Remark: $-\frac{1}{3} \leq \mu(\kappa) < 0$, spectrum <u>doubly degenerate</u> except for $\mu(0) = -\frac{1}{3}$

MOYAL REPRESENTATION OF SFT

AIM: Writing VSFT in terms of Moyal $*$ product

First, define

$$o_\kappa^+ = -\sqrt{2}\, i \sum_{n=1}^{\infty} v_{2n-1}(\kappa)\, a_{2n-1}^+$$

$$e_\kappa^+ = \sqrt{2} \sum_{n=1}^{\infty} v_{2n}(\kappa)\, a_{2n}^+$$

with inverses

$$a_{2n-1}^+ = \sqrt{2}\, i \int_0^\infty d\kappa\; v_{2n-1}(\kappa)\, o_\kappa^+$$

$$a_{2n}^+ = \sqrt{2} \int_0^\infty d\kappa\; v_{2n}(\kappa)\, e_\kappa^+$$

and commutators

$$[o_\kappa, o_{\kappa'}^+] = [e_\kappa, e_{\kappa'}^+] = \delta(\kappa-\kappa'), \qquad [o_\kappa, e_{\kappa'}^+] = [e_\kappa, o_{\kappa'}^+] = 0$$

The 3-strings vertex becomes

$$|V_3\rangle = \exp\Big[\int_0^\infty d\kappa \Big\{ -\frac{1}{2}\mu(\kappa)\left(o_\kappa^{(1)+} o_\kappa^{(1)+} + e_\kappa^{(1)+} e_\kappa^{(1)+} + \text{cyc.}\right)$$
$$-\frac{1}{2}\left(\mu^{12}(\kappa)+\mu^{21}(\kappa)\right)\left(o_\kappa^{(1)+} o_\kappa^{(2)+} + e_\kappa^{(1)+} e_\kappa^{(2)+} + \text{cyc.}\right)$$
$$-\frac{i}{2}\left(\mu^{12}(\kappa)-\mu^{21}(\kappa)\right)\left(e_\kappa^{(1)+} o_\kappa^{(2)+} - o_\kappa^{(1)+} e_\kappa^{(2)+} + \text{cyc.}\right)\Big\}\Big]|0\rangle$$

Now define combinations

$$\hat{x}_\kappa = \frac{i}{\sqrt{2}}\left(e_\kappa - e_\kappa^+\right) = \sqrt{2}\sum_{n=1}^{\infty} v_{2n}(\kappa)\sqrt{2n}\,\hat{x}_{2n}$$

$$\hat{y}_\kappa = \frac{i}{\sqrt{2}}\left(o_\kappa - o_\kappa^+\right) = -\sqrt{2}\sum_{n=1}^{\infty} \frac{v_{2n-1}(\kappa)}{\sqrt{2n-1}}\,\hat{p}_{2n-1}$$

there are also

$$\hat{z}_\kappa = \frac{1}{\sqrt{2}}\left(e_\kappa + e_\kappa^+\right)$$

$$\hat{w}_\kappa = \frac{1}{\sqrt{2}}\left(o_\kappa + o_\kappa^+\right)$$

The eigenvalues x_κ, y_κ

$$\hat{x}_\kappa |x_\kappa\rangle = x_\kappa |x_\kappa\rangle \quad , \quad \hat{y}_\kappa |y_\kappa\rangle = y_\kappa |y_\kappa\rangle$$

are the Moyal conjugate coordinates

$$\boxed{\left[x_\kappa , y_{\kappa'}\right]_* = i\,\theta_\kappa\,\delta(\kappa-\kappa')} \qquad \theta_\kappa = 2\tanh\frac{\pi\kappa}{4}$$

Moyal product for string fields:

$$|\Psi\rangle \dashrightarrow \underbrace{\Psi(\{x_{2n}\},\{x_{2n+1}\})}_{\langle x(\sigma)|\Psi\rangle} \dashrightarrow \tilde{\Psi}(\{x_{2n}\},\{p_{2n-1}\}) \dashrightarrow \Psi^M(x_\kappa, y_\kappa)$$

Then

$$\underbrace{|\Psi\rangle_1 * |\Psi\rangle_2}_{\text{Witten}} \longleftrightarrow \underbrace{\Psi_1^M * \Psi_2^M}_{\text{Moyal}}$$

Sliver takes form

$$|\Xi\rangle = \mathcal{N}^{26}\, e^{-\frac{1}{2}\int_0^\infty d\kappa\, \frac{\theta_\kappa - 2}{\theta_\kappa + 2}\left(e^+_\kappa e^+_\kappa + o^+_\kappa o^+_\kappa\right)}|0\rangle$$

One can switch on a background B field.

Ex.: along 24-th, 25-th directions $\quad B_{\alpha\beta} = \begin{pmatrix} 0 & B \\ -B & \end{pmatrix}$

$$\hookrightarrow \quad G_{\alpha\beta} = \sqrt{\mathrm{Det}\, G}\; \delta_{\alpha\beta} \qquad \mathrm{Det}\, G = \left(1 + (2\pi B)^2\right)^2$$

$$\theta^{\alpha\beta} = -(2\pi\alpha')^2 B\, \epsilon^{\alpha\beta}$$

The canonical commutators change:

$$\left[a_M^{(r)\alpha},\, a_N^{(s)\beta\dagger} \right] = G^{\alpha\beta}\, \delta_{MN}\, \delta^{rs}$$

The vertex change

$$V_{00} \longrightarrow \mathcal{V}_{00}^{\alpha\beta,rs} = G^{\alpha\beta}\delta^{rs} - \frac{2A^{-1}b}{2a^2+3}\left(G^{\alpha\beta}\phi^{rs} - i a\, \epsilon^{\alpha\beta}\chi^{rs}\right)$$

$$V_{0n} \longrightarrow \mathcal{V}_{0n}^{\alpha\beta,rs} = \frac{2A^{-1}\sqrt{b}}{4a^2+3}\sum_{t=1}^{3}\left(G^{\alpha\beta}\phi^{rt} - i a\, \epsilon^{\alpha\beta}\chi^{rt}\right)V_{0n}^{ts}$$

$$V_{mn} \longrightarrow \mathcal{V}_{mn}^{\alpha\beta,rs} = G^{\alpha\beta}V_{mn}^{rs} - \frac{2A^{-1}}{4a^2+3}\sum_{t,v=1}^{3} V_{m0}^{rv}\left(G^{\alpha\beta}\phi^{vt} - i a\, \epsilon^{\alpha\beta}\chi^{vt}\right)V_{0n}^{ts}$$

where

$$\phi = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} \qquad \chi = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad \begin{array}{l} \chi^2 = -2\phi \\ \phi\chi = \chi\phi = \frac{3}{2}\chi \\ \phi^2 = \frac{3}{2}\phi \end{array}$$

$$A = V_{00} + \frac{b}{2} \qquad a = -\frac{\pi^2}{A} B$$

Introduce $\quad C_{MN} = (-1)^N \delta_{MN}$

and define $\quad \mathcal{X}^{rs} \equiv C\, V^{rs} \qquad\qquad \mathcal{X}^{11} = \mathcal{X}$

then $\quad [\mathcal{X}^{rs}, \mathcal{X}^{r's'}] = 0$

and

$$|\mathcal{S}\rangle = \left(\mathrm{Det}(1-X)\,\mathrm{Det}(1+T)\right)^{12} e^{-\frac{1}{2}\eta_{\bar{\mu}\bar{\nu}} \sum_{m,n\geq 1} a_m^{\bar{\mu}+} S_{mn}\, a_n^{\bar{\nu}+}} |0\rangle \cdot$$

$$\cdot \frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}\,(\mathrm{Det}\,G)^{1/4}} \sqrt{\mathrm{Det}(1-\mathcal{X})\,\mathrm{Det}(1+\mathcal{T})}\; e^{-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha+} \mathcal{S}_{MN}^{\alpha\beta} a_N^{\beta+}} |\tilde{0}\rangle$$

$\bar{\mu}, \bar{\nu} = 0, \ldots, 24$

$\mathcal{S} = C\mathcal{T} \qquad\qquad \mathcal{T} = \frac{1}{2\mathcal{X}}\left(1+\mathcal{X} - \sqrt{(1+3\mathcal{X})(1-\mathcal{X})}\right)$

$\mathcal{T}$ is solution of

$$\mathcal{X}\mathcal{T}^2 - (1+\mathcal{X})\mathcal{T} + \mathcal{X} = 0$$

Then

$$|\mathcal{S}\rangle * |\mathcal{S}\rangle = |\mathcal{S}\rangle$$

and

$$\frac{e_{23}}{e_{25}} = \frac{(2\pi)^2}{\sqrt{[\text{illegible}]}^2}\, a \qquad \text{right ratio for D-brane tensions !}$$

$$a = \frac{A^4(3+4a^2)^2}{2\pi b^3(\mathrm{Det}\,G)^{1/4}} \frac{\mathrm{Det}(1-\mathcal{X})^{3/4}\,\mathrm{Det}(1+3\mathcal{X})^{1/4}}{\mathrm{Det}(1-X)^{[\text{illegible}]}\,\mathrm{Det}(1+3X)^{1/2}} = 1$$

Field theory limit: $\alpha' \to 0$

In this limit:

$$V_{00}^{\alpha\beta,rs} \longrightarrow G^{\alpha\beta}\delta^{rs} - \frac{4}{4a^2+3}\left(G^{\alpha\beta}\phi^{rs} - i a \epsilon^{\alpha\beta}\chi^{rs}\right)$$

$$V_{0n}^{\alpha\beta,rs} \longrightarrow 0$$

$$V_{mn}^{\alpha\beta,rs} \longrightarrow G^{\alpha\beta} V_{mn}^{rs}$$

Introducing

$$|x\rangle = \sqrt{\frac{2\sqrt{\det G}}{b\pi}}\; e^{-\frac{1}{b} x^\alpha G_{\alpha\beta} x^\beta - \frac{2}{\sqrt{b}} i a_0^{\alpha+} G_{\alpha\beta} x^\beta + \frac{1}{2} a_0^{\alpha+} G_{\alpha\beta} a_0^{\beta+}} |\Omega_b\rangle$$

One finds

$$\langle x | \mathcal{S} \rangle = \frac{1}{\pi} e^{-\frac{1}{2|a|b} x^\alpha G_{\alpha\beta} x^\beta} |\Xi\rangle$$

$$= \frac{1}{\pi} e^{-\frac{x^\alpha \delta_{\alpha\beta} x^\beta}{\theta}} |\Xi\rangle \qquad \theta = \frac{1}{B}$$

<u>More</u> <u>solutions</u> ?

Define projectors:

$$\rho_1 = \frac{\chi^{12}(1-\tau\chi) + \tau(\chi^{21})^2}{(1+\tau)(1-\chi)} \qquad \rho_2 = \frac{\chi^{21}(1-\tau\chi) + \tau(\chi^{12})^2}{(1+\tau)(1-\chi)}$$

$$\rho_1^2 = \rho_1 \qquad \rho_2^2 = \rho_2 \qquad \rho_1 + \rho_2 = 1$$

Now take two "vectors" ξ and ζ $\qquad \xi = \{\xi_{mn}\}$, $\zeta = \{\zeta_{mn}\}$

such that

$$\rho_1 \xi = 0, \quad \rho_2 \xi = \xi \qquad \rho_1 \zeta = 0, \quad \rho_2 \zeta = \zeta$$

Define

$$x = a^+ \tau \xi \; a^+ C \zeta \qquad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$|\Lambda_n\rangle = (-\alpha)^n \underbrace{L_n}_{\text{Laguerre pol.}}\left(\frac{x}{\alpha}\right) |S\rangle \qquad n = 0, 1, 2, \ldots$$

Moreover require

$$\xi \frac{1}{1-\tau^2} \zeta = -1, \qquad \xi \frac{\tau}{1-\tau^2} \zeta = -\alpha$$

α = const.

Then

$$\boxed{\begin{aligned} |\Lambda_n\rangle * |\Lambda_m\rangle &= \delta_{n,m}|\Lambda_n\rangle \\ \langle\Lambda_n|\Lambda_m\rangle &= \delta_{n,m}\langle\Lambda_0|\Lambda_0\rangle \end{aligned}}$$

Field theory limit

$$\langle x|\Lambda_n\rangle \longrightarrow \frac{1}{\pi}(-1)^n L_n\left(\frac{x^2+y^2}{\theta}\right) e^{-\frac{x^2+y^2}{\theta}} |\Xi\rangle$$

GMS solitons

Remarkable! Isomorphism between SFT $*$ product and Moyal product

$P_n = |n\rangle\langle n|$ $\qquad\qquad$ $\psi_n(x,y) = \frac{1}{\pi}(-1)^n L_n\left(\frac{x^2+y^2}{\theta}\right) e^{-\frac{x^2+y^2}{\theta}}$

$$|\Lambda_n\rangle \longleftrightarrow P_n \longleftrightarrow \psi_n$$

$$|\Lambda_n\rangle * |\Lambda_{n'}\rangle \longleftrightarrow P_n P_{n'} \longleftrightarrow \psi_n * \psi_{n'}$$

$$\langle\Lambda_n|\Lambda_{n'}\rangle \longleftrightarrow \mathrm{Tr}(P_n P_{n'}) \longleftrightarrow \int dx\,dy\, \psi_n \psi_{n'}$$