II SUMMER SCHOOL IN MODERN MATHEMATICAL PHYSICS

KOPAONIK 2002

Reductions and orbifolds: construction of self-dual 4-metrics

NOVICA BLAŽIĆ - FACULTY OF MATHEMATICS
UNIVERSITY OF BELGRADE
blazicn@matf.bg.ac.yu

1. Basic examples
2. **Problem.** What and why to construct.
3. How?
4. Properties of the metric, $O_{p,q}$.
5. Neutral signature $(--++)$
6. Curvature and a local dynamics of $O_{p,q}$

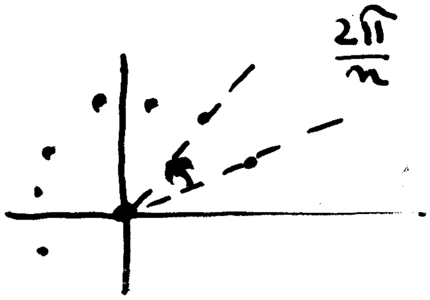
0

1. Basic examples

- manifold
- orbit spaces

Ex. 1. $\mathbb{C}$, $Z_n = \{ e^{2\pi i \frac{k}{n}} \mid 0 \leq k \leq n-1 \} \ni \tau$

$$z \to \tau z$$



- orbifold (locally) - manifold deceided by the finite group of $Isom(M, g)$

- simpliest singularities

Ex. 2. (projective space)

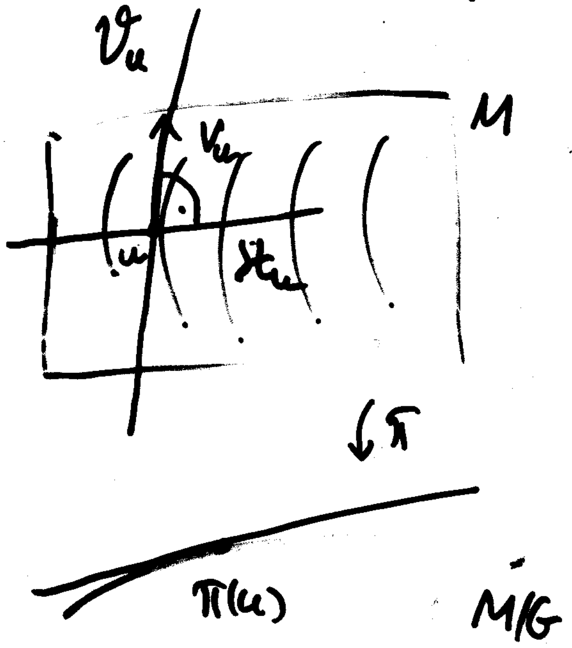
$$\begin{array}{c} \mathbb{C}^{n+1} \\ \cup \\ S^1 \to S^{2n+1} \\ \downarrow \\ S^{2n+1}/S^1 = \mathbb{CP}^n \end{array}$$

$G = S^1 = U(1)$

$z \to e^{2\pi i t} z$

free action!

M, $G \leq \mathrm{Isom}(M,g)$, compact



free action

M/G is manifold

- metric

Ex.3 (Weighted projective space)

$\mathbb{C}^3 \supset S^5$

$\downarrow$

$\mathbb{C}P^2_{p,q,r}$

$(p,q,r)=1$, $p,q,r \in \mathbb{N}$

S^1-action, $\tau \in S^1$

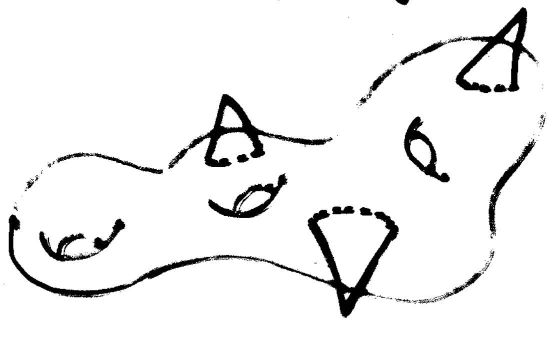
$(z_0, z_1, z_2) \in S^5$

$\downarrow$

$(\tau^p z_0, \tau^q z_1, \tau^r z_2)$

not free but loc. free! orbifold

- true in a general case



2

2. Problem:

$\boxed{\text{Construct self-dual, Einstein, compact 4-dim metrics?}}$

+ <u>non-symm</u>.

- physics motivation

$f: \Sigma \to X_{\text{4-dim}}$ | classical σ-models admits (locally) an $N=2$ supersym. ext iff X is self-dual, $\mathcal{E}$, scalar $\neq 0$

Breitenlohner, Sohnius, '81 $\quad \dfrac{SU(n+1)}{SU(n-1) \times SU(2) \times U(1)}$

- math. motivation

twistor space

$$Z \ \text{cplx, } 3$$
$$\downarrow$$
$$M$$

- self duality, def
(half-conformally flat)

$$\begin{array}{cc} \Lambda^2_+ & \Lambda^2_- \end{array}$$

$$R = \left[\begin{array}{c|c} w_+ & B \\ \hline B^* & w_- \end{array}\right] \qquad R_+ \doteq \lambda\, Id.$$

- results
Hitchin, Friedric, Kürke '81

cpct mnfld, scal. > 0, self-dual ⇒ S^4, $\mathbb{C}P^2$?

- Ricci flat
  - K3 - surface
  - Eguchi-Hanson metric (hK)
  - Taub-Nut

  non-symmetric

- non-Ricci-flat

3. How? — by reduction

Galicki, Lawson, Math. Ann., 1987

self-dual $\overset{4}{\mathcal{E}}$ ⟵ (quaternionic kähler (qK) geometry) ⟶ reductions

---

$\mathbb{H} = \{a + bi + cj + dk \mid i^2 = j^2 = k^2 = -1, \; ij = -ji = k, \dots\}$

$|uv|^2 = |u|^2 |v|^2 \qquad \mathbb{H}_1 = S^3$

$\{J_1, J_2, J_3\}, \quad \omega_i = \ldots (J_1, J_2, J_3)$

$J_i \circ J_j = -\delta_{ij}\, \mathrm{id} + \varepsilon_{ijk} J_k$

- ~~similar cur~~ $Hol \subset Sp_1 \times Sp_m / \mathbb{Z}_2 = Sp_1 Sp_m$
  - J_i – … m.
- $\omega_i(V, W) = \langle J_i V, W \rangle \quad \Omega = \sum \omega_i \wedge \omega_i$
  $\nabla \Omega = 0 \iff q.K.$ (h.K. $D\omega_i = 0$) glob. def.
- similar curvature tensors
- quaternionic projective space $\mathbb{H}P^n$ (right)
  $(u_0, \dots, u_n) \sim (u_0 t, \dots, u_n t), \quad t \in \mathbb{H}_1 = S^3$

$(\mathbb{H}^{n+1} - \{0\}) / \mathbb{H}_1 = S^{4n+3}/S^3 \longrightarrow \mathbb{H}P^n \qquad S^{4n+3} \leftarrow S^3$, $\downarrow$ $\mathbb{H}P^n$

M + [redacted] symm.) ?] by G

$$
\begin{tikzcd}
G \arrow[r] & M \\
G \arrow[r] & M_0 \arrow[u, hook] \arrow[d] \\
 & M_0/G
\end{tikzcd}
$$

M - quaternionic Kähler (qK)

$Hol \subset Sp_1 \cdot Sp_n$

Table 1. HyperKähler quotients of flat spaces

Manifold M	Quotient group G_p	$\dim \tilde{M}_p$	HyperKähler metric on $\tilde{M}_p$
$\mathbf{H}^n$	$U(1)^{n-1}$	4	Multi-Eguchi-Hanson gravitational instanton
$\mathbf{H}^{n-1} \times \mathbf{R}^3 \times S^1$	$U(1)^{n-1}$	4	Multi-Taub-NUT gravitational instanton
$\mathbf{H}^n$	$U(1)$	$4n-4$	Calabi metrics on $T^*(\mathbb{C}P(n-1))$
$\mathbf{H}^{n-1} \times \mathbf{R}^3 \times S^1$	$U(1)$	$4n-4$	Lindström-Roček metrics: new generalization with Taubian infinity
$\mathbf{H}^n$	$U(1)^m$	$4n-4m$	Multi-Calabi spaces
$\mathbf{H}^{m(n+m)}$	$U(m)$	$4nm$	Lindström-Roček spaces: hyperKähler metrics on cotangent bundles of complex Grassmannians

~~unicepral~~ character

07

$(p,q)=1,\ p,q\in\mathbb{Z}^+$

$$\mathbb{R}^{12}=\mathbb{H}^3$$
$$\cup$$
$$S^{11}\leftarrow S^3$$
$$\downarrow$$
$$S^1 \xrightarrow{\varphi_{p,q}} \mathbb{H}P^2 = S^{11}/S^3$$
$$\cup$$
$$S^1 \xrightarrow{\varphi_{p,q}} \mathcal{L}_{\mathbb{H}} \quad \text{loc. free}$$
$$\downarrow$$
$$\mathcal{L}_{\mathbb{H}}/S^1 = \boxed{\mathcal{O}_{q,p}}$$

$$(u_0, u_1, u_2)$$
$$\downarrow$$
$$(e^{2\pi i q t}\cdot u_0,\ e^{2\pi i p t}\cdot u_1,\ e^{2\pi i p t}\cdot u_2)$$

$\mathcal{L}_{\mathbb{H}}$:

$$q\bar{u}_0 i u_0 + p\bar{u}_1 i u_1 + p\bar{u}_2 i u_2 = 0$$

5. Neutral signature

$(--++)$, $(2,2)$

Srđan Vukmirović, N.B. math ArXiv 0206081

vsrdjan@matf.bg.ac.yu

blazicn@

– Ooguri-Vafa, Lu-Pope-Sezgin, Kamada-Machida, Barret-Gibbons ... ,

$$SO(2,2) = SL(2) \times SL(2)$$

$$\mathbb{H} \longrightarrow \widetilde{\mathbb{H}} = C(1,1) = C(0,2)$$

Clifford algebra

$$i^2=-1,\quad j^2=k^2=+1,\quad ij=k=-ji$$

Paraquaternionic numb.

$$q = x+yi+zj+wk$$

$$\bar{q} = x-yi-zj-wk$$

(2,2)

$$|q|^2 = q\bar{q} = x^2+y^2-z^2-w^2$$

natural language

$$|q_1 q_2|^2 = |q_1|^2 |q_2|^2$$

$$S^{2,1} = \widetilde{\mathbb{H}}_1 = \{q \in \widetilde{\mathbb{H}} : |q|^2 = 1\} = SU(1,1)$$

$$\begin{array}{c} \tilde{\mathbb{H}}^3 = \mathbb{R}^{6,6} \\ \cup \\ S^{6,5} \leftarrow S^{2,1} \\ \downarrow \pi \\ \mathbb{R} \xrightarrow{\varphi_{p,q}} \tilde{\mathbb{H}}P^2 \\ \cup \\ \mathbb{R} \xrightarrow{\varphi_{p,q}} \mathcal{K}_0 \quad \text{free action!} \\ \cup \\ \mathbb{R} \rightarrow \mathcal{K} \quad |V_u|^2 = 0 \\ \downarrow \\ \mathcal{K}/\mathbb{R} = \boxed{\mathcal{O}_{q,p}} \end{array}$$

$$(u_0, u_1, u_2) \xrightarrow{\varphi_{p,q}} (e^{jqt}u_0, e^{jpt}u_1, e^{jpt}u_2)$$

$\mathcal{K}_0$:

$$q\bar{u}_0 j u_0 + p\bar{u}_1 j u_1 + p\bar{u}_2 j u_2 = 0$$

$$V_u = j(qu_0, pu_1, pu_2)$$

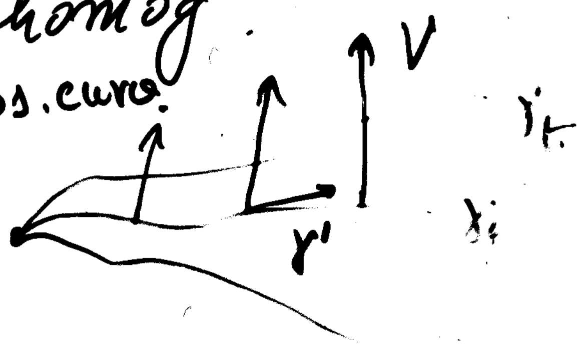
metric nim 7

– (2,2) self-dual, E

6. Curvature of $O_{q,p}$

yes

- self-dual
- loc. dynamically homog.
- pos. curv.



not

- symmetric, $DR \neq 0$
homogenous,

Jacobi

$$\boxed{V'' = -R(V,\gamma')\gamma' = J_{\gamma'}(V)}$$

$$V \longrightarrow J_{\gamma'}(V)$$

$$\lambda_1 = \lambda_2 = 4 - \frac{2p^4q^2}{|V_u|^6}, \quad \lambda_3 = 4 + \frac{4p^4q^2}{|V_u|^6}$$

$l_u = i(q l_0, p l_1, p l_2)$

$|V_u|$ - depends on a point but not on a direct.

- positive sectional curve.