THE TELEPARALLEL THEORY OF GRAVITY

M. Blagojević
Kopaonik '02

1. Poincaré gauge theory
2. The teleparallel theory
3. Topological teleparallel 3d gravity

① POINCARÉ GAUGE THEORY

A. Global Poincaré symmetry

1. We assume that spacetime has the structure of M_4 (at least locally).

$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$ Metric, in an inertial ref. frame

Isometry of M_4:

$x'^\mu = x^\mu + \xi^\mu(x)$, Killing eq. $\Rightarrow$

$$\underline{\xi^\mu = \omega^\mu{}_\nu x^\nu + \varepsilon^\mu} \qquad (1)$$

$K = \xi^\mu \partial_\mu$, Killing vectors of the global Poincare transformations

2. Tangent space

T_P, the choice of basis not unique

Coordinate frame: $\vec{e}_\mu$, $\vec{e}_\mu \cdot \vec{e}_\nu = g_{\mu\nu}$

Local Lorentz frame: $\vec{e}_i$, $\vec{e}_i \cdot \vec{e}_j = \eta_{ij}$

(vierbein, tetrad)

$\vec{e}_i = e_i{}^\mu \vec{e}_\mu$, $\vec{e}_\mu = e^i{}_\mu \vec{e}_i$

$dx^i = e^i{}_\mu dx^\mu$ local Lorentz coordinates

x^μ = inertial $\Rightarrow$ we can choose $\vec{e}_i$ so that $\vec{e}_i = \delta_i^\mu \vec{e}_\mu$.

3. <u>Matter field</u> $\phi(x)$

scalar, spinor, ... defined in T_P, with respect to an L frame.

$$\delta_0\phi = \Big[\tfrac{1}{2}\omega^{ij}\underbrace{(x_i\partial_j - x_j\partial_i + \Sigma_{ij})}_{M_{ij}} + \varepsilon^{\mu}\underbrace{(-\partial_\mu)}_{P_\mu}\Big]\phi \equiv \mathcal{P}\phi \quad (2)$$

Σ_{ij}: spin matrix

(M_{ij}, P_μ) Poincaré generators in the space of fields defined in T_P.

4. <u>global Poincaré invariance</u>

$$I = \int d^4x\ \mathcal{L}_M(\phi, \partial_K\phi)$$

The action integral is invariant under global Poincare transf's, $x' = x + \xi$, if

$$\Delta\mathcal{L} \equiv \delta_0\mathcal{L} + \partial_\mu(\xi^\mu\mathcal{L}) = 0 \qquad (3)$$

where

$$\delta_0\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta_0\phi + \frac{\partial\mathcal{L}}{\partial\,\partial_K\phi}\delta_0\partial_K\phi$$

$$\delta_0\phi = (\tfrac{1}{2}\omega\cdot M + \varepsilon\cdot P)\phi = \mathcal{P}\phi$$

$$\delta_0\partial_K\phi = \partial_K\delta_0\phi = \mathcal{P}\partial_K\phi + \omega_K{}^i\partial_i\phi \equiv \mathcal{P}_K{}^i\partial_i\phi$$

Comments:

- In the derivation of (3), we assumed $\delta_0\eta = 0$
- We can allow $\Delta\mathcal{L} = \partial_\mu\Lambda^\mu$, as the surface term in the action does not influence field eqs.

5. Noether theorem

Global Poincaré inv. $\Rightarrow$ $\partial_\mu J^\mu = 0$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta_0 \phi + \xi^\mu \mathcal{L} = \frac{1}{2} \omega^{\nu\rho} M^\mu{}_{\nu\rho} - \varepsilon^\nu T^\mu{}_\nu$$

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$$ canonical energy-mom.

$$M^\mu{}_{\nu\rho} = (x_\nu T^\mu{}_\rho - x_\rho T^\mu{}_\nu) - S^\mu{}_{\nu\rho}$$ ang. mom.

$$\partial_\mu T^\mu{}_\nu = 0 \qquad \partial_\mu M^\mu{}_{\nu\rho} = T_{\nu\rho} - T_{\rho\nu} - \partial_\mu S^\mu{}_{\nu\rho} = 0$$

B. Localization of Poincaré symmetry

1. Start with a matter Lagrangian $\mathcal{L}_M(\phi, \partial\phi)$ invariant under $P(1,3)$:

$$\Delta \mathcal{L}_M = 0$$

localization: constant $\varepsilon^i \rightarrow \varepsilon^i(\underline{x})$
constant $\omega^i{}_k \rightarrow \omega^i{}_k(\underline{x})$.

The invariance condition $\Delta \mathcal{L}_M = 0$ is now violated because

(i) $\delta_0 \partial_k \phi = \xi_k{}^i \partial_i \phi + (\partial\omega, \partial\varepsilon)$-terms
$\neq$ the old transformation rule for $\partial_k \phi$.

(ii) the term $\partial_\mu \xi^\mu \cdot \mathcal{L}_M$ in $\Delta \mathcal{L}_M$ is different from zero.

$$\partial_\mu \xi^\mu = (\partial_\mu \omega^\mu{}_\nu) x^\nu + \partial_\mu \varepsilon^\mu \neq 0.$$

$$\Delta \mathcal{L}_M = \tfrac{1}{2} (\partial_\mu \omega^{ij}) S^\mu{}_{ij} - (\partial_\mu \xi^i - \omega^i{}_\mu) T^\mu{}_i \neq 0.$$

2. The violation of local invariance can be <u>compensated</u> by certain modifications of the original theory.

(i) $\mathcal{L}_M(\phi, \partial_k \phi) \rightarrow \mathcal{L}'_M \equiv \mathcal{L}_M(\phi, \nabla_k \phi)$

∇_k = covariant derivative; $\nabla_k \phi$ transforms according to the "old rule":

$$\underline{\delta_0 \nabla_k \phi = \mathcal{P} \nabla_k \phi + \omega_k{}^i \nabla_i \phi} \qquad (5)$$

Construction of $\nabla_k \phi$ with the help of <u>compensating fields</u>:

a) $\nabla_\mu \phi = (\partial_\mu + \underline{A_\mu}) \phi$, $\quad A_\mu = \tfrac{1}{2} A^{ij}{}_\mu \Sigma_{ij}$

b) $\nabla_k \phi = \delta_k^\mu \nabla_\mu \phi - \underline{A_k{}^\mu} \nabla_\mu \phi \equiv h_k{}^\mu \nabla_\mu \phi$

Condition (5) $\Rightarrow$

$$\underline{\delta_0 A^{ij}{}_\mu = -\nabla_\mu \omega^{ij} - \partial_\mu \xi^\rho \cdot A^{ij}{}_\rho - \xi \cdot \partial A^{ij}{}_\mu} \qquad (6a)$$

$$\underline{\delta_0 h_i{}^\mu = \omega_i{}^k h_k{}^\mu + \partial_\rho \xi^\mu h_i{}^\rho - \xi \cdot \partial h_i{}^\mu} \qquad (6b)$$

(ii) Modification of $\mathcal{L}_M$

$$\mathcal{L}_M' = \mathcal{L}_M(\phi, \nabla_K \phi) \,, \qquad \delta_0 \mathcal{L}_M' + \xi \cdot \partial \mathcal{L}_M' = 0$$

$$\tilde{\mathcal{L}}_M = \Lambda \, \mathcal{L}_M(\phi, \nabla_K \phi)$$

a function of new fields

$$\Delta \tilde{\mathcal{L}}_M = \delta_0 \tilde{\mathcal{L}}_M + \partial_\mu(\xi^\mu \tilde{\mathcal{L}}_M) = 0$$

if $\delta_0 \Lambda + \partial_\mu(\xi^\mu \Lambda) = 0$

simple solution: $\Lambda = b = \det^{-1}(h_i{}^\mu)$

Define $b^i{}_\mu$, the dual of $h_i{}^\nu$: $b^i{}_\mu h_i{}^\nu = \delta_\mu^\nu$

$$\underline{\delta_0 b^i{}_\mu = \omega^i{}_K b^K{}_\mu - \partial_\mu \xi^\rho . b^i{}_\rho - \xi \cdot \partial b^i{}_\mu} \qquad (7)$$

$\Rightarrow$ The new Lagrangian, invariant under local $P(1,3)$, has the form

$$\underline{\tilde{\mathcal{L}}_M = b \, \mathcal{L}_M(\phi, \nabla_K \phi)}$$

__Comment__. In order to have a clear geom. interpretation, it is convenient to treat

$i, j, k, \ldots$ as local Lorentz indices

$\mu, \nu, \rho, \ldots$ as coordinate indices.

3. __Field strengths__

$$[\nabla_K, \nabla_\ell]\phi = \tfrac{1}{2} F^{ij}{}_{K\ell} \Sigma_{ij} \phi - F^s{}_{K\ell} \nabla_s \phi$$

$$F^i{}_{\mu\nu} = \nabla_\mu b^i{}_\nu - \nabla_\nu b^i{}_\mu \qquad \text{translational}$$

$$F^{ij}{}_{\mu\nu} = \partial_\mu A^{ij}{}_\nu + A^i{}_{s\mu} A^{sj}{}_\nu - (\mu \leftrightarrow \nu) \qquad \text{Lorentz}$$

$$F^i{}_{jk} = h_j{}^\mu h_k{}^\nu F^i{}_{\mu\nu}, \qquad F^{ij}{}_{kl} = h_k{}^\mu h_l{}^\nu F^{ij}{}_{\mu\nu}.$$

4. Bianchi identities

$$\varepsilon^{\lambda\mu\nu\rho} \nabla_\mu F^i{}_{\nu\rho} = \varepsilon^{\lambda\mu\nu\rho} F^i{}_{s\nu\rho} b^s{}_\mu \qquad (1st)$$

$$\varepsilon^{\lambda\mu\nu\rho} \nabla_\mu F^{ij}{}_{\nu\rho} = 0 \qquad (2nd)$$

5. Complete Lagrangian

$$\tilde{\mathcal{L}} = b\mathcal{L}_M(\phi, \nabla_k \phi) + b\mathcal{L}_F(F^{ij}{}_{mn}, F^i{}_{mn}) \qquad (9)$$

6. Generalized conservation laws

$$\tilde{\mathcal{L}}_M \qquad \tau^\mu{}_k = -\frac{\delta \tilde{\mathcal{L}}_M}{\delta b^k{}_\mu} \qquad (10a)$$

$$\sigma^\mu{}_{ij} = -\frac{\delta \tilde{\mathcal{L}}_M}{\delta A^{ij}{}_\mu} \qquad (10b)$$

dynamical currents

$$b^i{}_\mu \nabla_\rho \tau^\rho{}_k = \tau^\rho{}_k F^k{}_{\mu\rho} + \tfrac{1}{2} \sigma^\rho{}_{ij} F^{ij}{}_{\mu\rho}$$

$$\nabla_\mu \sigma^\mu{}_{ij} = \tau_{ij} - \tau_{ji}$$

C. Geometric and gauge structure of PGT

1. In PGT we have

compensating fields	$b^i{}_\mu,\ A^{ij}{}_\mu$
covariant derivative	$\nabla_K = h_K{}^\mu(\partial_\mu + A_\mu)$
field strengths	$F^i{}_{\mu\nu},\ F^{ij}{}_{\mu\nu}$

This theory can be tought of as a field theory in M_4. However, it would be unnatural to ignore strong geometric analogies.

- $\nabla_\mu \phi$ can be interpreted as the geometric covariant derivative $D_\mu\phi = (\partial_\mu + \omega_\mu)\phi$ since
  - $\nabla_\mu\phi$ has an additional dual index, as compared to ϕ
  - ∇_μ act linearly, obeys Leibnitz rule, commutes with contraction and $\nabla_\mu f = \partial_\mu f$ for f = scalar.
- By comparing ∇_μ and $D_\mu \Rightarrow A_\mu = \omega_\mu$
- $b^i{}_\mu = e^i{}_\mu$ on the basis of its transf. porpert's.
- Local Lorentz symmetry $\Rightarrow$ <u>metricity cond.</u>

$$\nabla_\mu \eta_{ij} = \partial_\mu \eta_{ij} + A_i{}^s{}_\mu \eta_{sj} + A_j{}^s{}_\mu \eta_{is}$$
$$= A_{ij\mu} + A_{ji\mu} = 0$$

$\Rightarrow$ <u>PGT has the geometric structure of the Riemann-Cartan spacetime.</u>

2. <u>PGT does not have the structure of an "ordinary" gauge theory.</u>

$$A_\mu = e^i{}_\mu P_i + \tfrac{1}{2}\,\omega^{ij}{}_\mu M_{ij}$$

$$u = u^i P_i + \tfrac{1}{2} u^{ij} M_{ij}$$

$$\delta_0 A_\mu = -\nabla_\mu u = -\partial_\mu u - [A_\mu, u]$$

Translations: $\delta_0 e^i{}_\mu = -\nabla'_\mu u^i$, $\quad \delta_0 \omega^{ij}{}_\mu = 0$

Lor. rotations: $\delta_0 e^i{}_\mu = u^i{}_k e^k{}_\mu$, $\quad \delta_0 \omega^{ij}{}_\mu = -\nabla'_\mu u^{ij}$

diff. from PGT

<u>Example</u>: Einstein-Cartan theory

$$I_{EC} = -a \int d^4x\, b R = a \tfrac{1}{2} \int d^4x\, \epsilon^{\mu\nu\lambda\rho}_{ijkl}\, b^k{}_\lambda b^l{}_\rho R^{ij}{}_{\mu\nu}$$

$$\delta_0^T I_{EC} = \frac{a}{2} \int d^4x\, \epsilon^{\mu\nu\lambda\rho}_{ijkl} \cdot 2\, b^k{}_\lambda \cdot (-\nabla_\rho u^l) \cdot R^{ij}{}_{\mu\nu}$$

$$= \frac{a}{2} \int d^4x\, \epsilon^{\mu\nu\lambda\rho}_{ijkl} \cdot T^k{}_{\rho\lambda} \cdot u^l \cdot R^{ij}{}_{\mu\nu}$$

$$+ a \int d^4x\, \epsilon^{\mu\nu\lambda\rho}_{ijkl}\, b^k{}_\lambda\, \nabla_\rho R^{ij}{}_{\mu\nu} \quad \neq 0$$

the second term vanishes (Bianchi id), but the first one remains $\neq 0$!!

Appendix: Geometric classification of spaces

Spacetime is often described as "4d continuum".
Topological space allows a precise formulation of the idea of continuity.

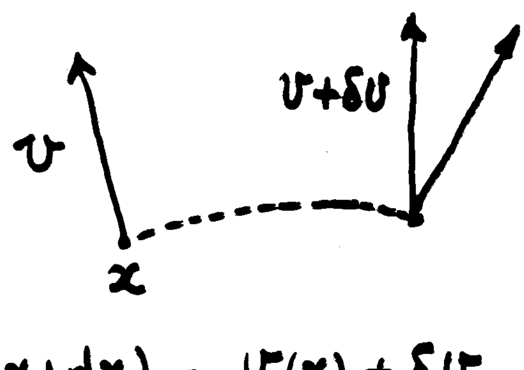
Differentiable manifold is defined as a topological space X which
- locally "looks like" an open subset of R^n, i.e. we can introduce local coordinates, and
- local coordinate systems are compatible (diff. transition functions).

Tangent space at P, T_P : vectors, tensors.
Dual of T_P : forms

The metric tensor : (0,2), symmetric nondegenerate tensor field

$$g : (u, v) \to u \cdot v = g_{\mu\nu} u^\mu u^\nu$$

Connection (parallel transport)

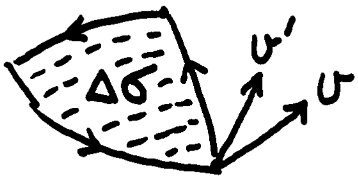


$$v_{PT}(x+dx) = v(x) + \delta v, \qquad \delta v^\mu = -\Gamma^\mu_{\lambda\rho} v^\lambda dx^\rho$$

$$Dv = v(x+dx) - v_{PT}(x+dx) \qquad \text{cov. derivative}$$

Torsion $\quad T^{\mu}{}_{\nu\rho} = \Gamma^{\mu}_{\nu\rho} - \Gamma^{\mu}_{\rho\nu}$

Curvature $\quad R^{\mu}{}_{\nu\rho\lambda} = \partial_\rho \Gamma^{\mu}_{\nu\lambda} + \Gamma^{\mu}_{\sigma\rho}\Gamma^{\sigma}_{\nu\lambda} - (\mu \leftrightarrow \nu)$



$$\Delta v^{\mu} = -\tfrac{1}{2} R^{\mu}{}_{\nu\lambda\rho}\, v^{\nu} \Delta\sigma^{\lambda\rho}$$

$$\begin{tikzcd}
 & & (L_4, g) \arrow[dll] & \\
Y_4 \arrow[dr, "Q=0"'] & & & \\
 & U_4 \arrow[dl, "T=0"'] \arrow[dr, "R=0"] & & \\
V_4 \arrow[dr, "R=0"'] & & T_4 \arrow[dl, "T=0"] & \\
 & M_4 & &
\end{tikzcd}$$

(L_4, g): Linarly connected, metric and connection independent

Y_4: Weil- Cartan space $\quad Q_{\mu\nu\rho} \equiv -\nabla_\mu(\Gamma) g_{\nu\rho} = -\varphi_\mu g_{\nu\rho}$

U_4: <u>Riemann-Cartan</u> space $\quad \nabla_\mu g_{\nu\rho} = 0$

V_4: R–mann

T_4: Weitzenböck <u>teleparallel</u>

M_4: Minkowski

<u>PGT has the geometric structure of the Riemann - Cartan space U_4.</u>

② THE TELEPARALLEL THEORY

Introduction

General geometric arena for PGT = U_4
Riemann-Cartan

$T \to 0$, $U_4 \to V_4$ = Riemannian space of GR

$R \to 0$, $U_4 \to T_4$ = teleparallel space

$$R^{ij}{}_{\mu\nu}(A) = 0 \qquad (1)$$

In T_4, parallel transport is path-independent (if some topological restrictions are adopted)

$\Rightarrow$ we have absolute parallelism

Physical interpretation:

There is a one-parameter family of TP Lagrangians which is empirically equiv. to GR

The Lagrangian

$$\tilde{\mathcal{L}} = b\mathcal{L}_T + \lambda_{ij}{}^{\mu\nu} R^{ij}{}_{\mu\nu} + \tilde{\mathcal{L}}_M$$

$$\mathcal{L}_T = a\left(A\, T_{ijk}T^{ijk} + B\, T_{ijk}T^{jik} + C\, T_i T^i\right) \equiv \beta\cdot T$$

$\lambda_{ij}{}^{\mu\nu}$ = Lagrange multipliers $\Rightarrow$ $R^{ij}{}_{\mu\nu} = 0$

$a = 1/2\varkappa$, A, B, C = free parameters

(i) $2A + B + C = 0$, $C = -1$

the one-parameter TP theory, gives the same results as GR in the linear, weak-field approx.

$\Rightarrow$ <u>empirically equivalent to GR</u>!

(ii) $2A - B = 0$ + (i) $[A = \frac{1}{4},\ B = \frac{1}{2},\ C = -1]$

this choice gives a TP theory <u>equivalent with GR</u> in the gravitational sector, but with <u>different interaction</u>

$A = \Delta + K$ Δ = Riemannian connection
K = contortion

$$\underset{\overset{\shortparallel}{0}}{abR(A)} = \underset{\overset{\uparrow}{GR}}{abR(\Delta)} + \tilde{\mathcal{L}}_T^{\shortparallel} - 2\partial_\mu(bT^\nu)\cdot a$$

teleparallel $\tilde{\mathcal{L}}_T$ with
$A = 1/4$, $B = 1/2$, $C = -1$

$$\Rightarrow \quad \underline{\tilde{\mathcal{L}}_T^{\shortparallel} = -\underset{\overset{\uparrow}{\text{GR Lagrangian!}}}{abR(\Delta)} + \text{div}}$$

Interaction

GR (V_4) : $\nabla = \partial +$ <u>Christoffel</u>

TP (T_4) : $\nabla = \partial + A$, A is a <u>pure gauge</u> since $R^{ij}{}_{\mu\nu}(A) = 0$

Field eqs

$$4\,\nabla_\rho(\tilde{\beta}_i{}^{\mu\rho}) - 4\tilde{\beta}^{nm\mu}T_{nmi} + h_i{}^\mu \tilde{\mathcal{L}}_T = \tau^\mu{}_i \qquad (3a)$$

$$4\,\nabla_\rho\,\lambda_{ij}{}^{\mu\rho} - 8\tilde{\beta}_{[ij]}{}^\mu = \sigma^\mu{}_{ij} \qquad (3b)$$

$$R^{ij}{}_{\mu\nu}(A) = 0 \qquad (3c)$$

(3c) defines TP geometry in PGT

(3a) is a dynamical eq. for $b^i{}_\mu$

symm. piece – analogous to Einstein's eq. in GR

antisymm. piece:

$$\nabla_\rho\,\tilde{\beta}_{[ij]}{}^\rho = \tau_{[ji]}$$

(3b) serves to determine $\lambda_{ij}{}^{\mu\nu}$

No. of eqs = $6\times 4 = 24$

∇ (3b) $\Rightarrow$ 6 identities

$$-8\,\nabla_\mu\,\tilde{\beta}_{[ij]}{}^\mu = \nabla_\mu\,\sigma^\mu{}_{ij} \underset{\uparrow}{=} \tau_{ij} - \tau_{ji}$$

angul. momentum conserv.

the resulting eq. coincides with antisymm. (3a)

$\Rightarrow$ No. of independent eqs = $24 - 6 = \underline{18}$

No. of $\lambda_{ij}{}^{\mu\nu} = 6\times 6 = \underline{36}$!

We shall see that No. of independent λ's is $\underline{18}$!

The λ symmetry

The TP Lagrangian (2) is inv. under local Poincaré transf's. In addition to that, it is also inv. under

$$\underline{\delta_0 \lambda_{ij}{}^{\mu\nu} = \nabla_\rho \varepsilon_{ij}{}^{\mu\nu\rho}} \tag{4}$$

antisymm. in (μ,ν,ρ)

proof:

$$\delta_0 \int d^4x \, \underline{\lambda_{ij}}{}^{\mu\nu} R^{ij}{}_{\mu\nu} = \int d^4x \, \underline{\nabla_\rho \varepsilon_{ij}{}^{\mu\nu\rho}} \cdot R^{ij}{}_{\mu\nu\rho}$$

$$= \text{surface term} - \int d^4x \, \underbrace{\varepsilon_{ij}{}^{\mu\nu\rho} \, \nabla_\rho R^{ij}{}_{\mu\nu}}_{}$$

$= 0$ by the 2nd Bianchi id.:
$\nabla_\rho R^{ij}{}_{\mu\nu} + \text{cicl}\,(\mu,\nu,\rho) = 0$

No. of gauge parameters $\varepsilon_{ij}{}^{\mu\nu\rho} = 6 \times 4 = \underline{24}$

However, canonical analysis shows that gauge transformations defined by $\underline{\varepsilon_{ij}{}^{\alpha\beta\gamma}}$ are <u>not independent</u> $\Rightarrow$ six parameters $\varepsilon_{ij}{}^{\alpha\beta\gamma}$ can be completly <u>discarded</u>!

No. of independent $\varepsilon_{ij}{}^{\mu\nu\rho} = 24 - 6 = 18$

$\Rightarrow$ No. of gauge-independent λ's $= 36 - 18 = \underline{18}$

<u>These λ's are completly determined by the (18) second field eqs. (3b).</u>

Orthonormal-teleparallel frames

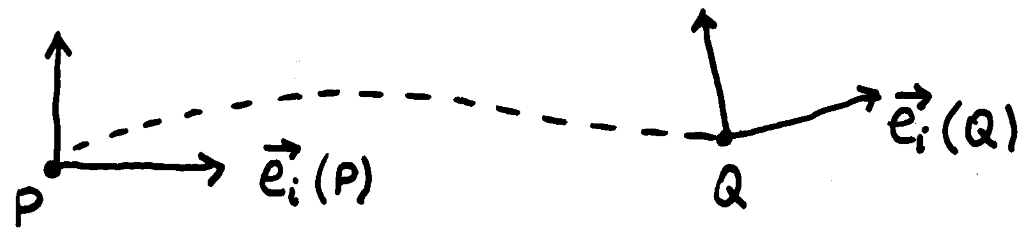
TP theories in U_4: $\qquad R^{ij}{}_{\mu\nu}(A) = 0 \qquad (1')$

Choose $\vec{e}_i$ at P, $\quad \vec{e}_i \cdot \vec{e}_j = \eta_{ij}$

if the manifold is parallelizable, i.e.

(1′) + some topological assumptions

$\Rightarrow$ parallel transport is path-independent



the resulting tetrad field $\vec{e}_i(x)$ is globally well defined

it is orthonormal and teleparallel $\Rightarrow$ OT frame.

$\vec{e}_i(Q)$ is parallel to $\vec{e}_i(P) \Rightarrow \quad A^{ij}{}_\mu = 0 \quad (5)$

In an OT frame, $\nabla = \partial$.

Eq. (5) defines a particular solution of (1′).

General solution:

$$\bar{A}^{ij}{}_\mu = \Lambda^i{}_m \Lambda^j{}_n \underset{\overset{\shortparallel}{0}}{A^{mn}{}_\mu} + \Lambda^i{}_m \partial_\mu \Lambda^{jm} = \Lambda^i{}_m \partial_\mu \Lambda^{jm}$$

$\Lambda^i{}_m$ – local Lorentz transf.

Thus, the choice $A^{ij}{}_\mu = 0$ breaks local Lorentz inv., but is compatible with $\Lambda^i{}_m = \text{const.}$

One can impose the gauge condition (5) directly in the TP action:

$$A^{ij}{}_\mu = 0\,, \qquad b^i{}_\mu = \text{dynamical}$$

The resulting theory = <u>translational gauge th.</u>

<u>$GR_{\parallel}$: the teleparallel form of GR</u>

$A = \frac{1}{4}\,,\; B = \frac{1}{2}\,,\; C = -1$

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_T^{\parallel} + \lambda_{ij}{}^{\mu\nu} R^{ij}{}_{\mu\nu} + \tilde{\mathcal{L}}_M \tag{6a}$$

$$0 = ab\,R(A) = ab\,R(\Delta) + \tilde{\mathcal{L}}_T^{\parallel} + \text{div.}$$

$$\tilde{\mathcal{L}} \to -ab\,R(\Delta) + \tilde{\lambda}_{ij}{}^{\mu\nu} R^{ij}{}_{\mu\nu} + \tilde{\mathcal{L}}_M \tag{6b}$$

1st field eq.

$$b\left[R_{ij}(\Delta) - \tfrac{1}{2}\eta_{ij} R(\Delta)\right] = \tau_{ji}/2a$$

$$\tau_{ij} = \tau_{ji} \quad \text{for consistency}$$

2nd field eq.

$$\nabla_\rho\left(4\lambda_{ij}{}^{\mu\rho} + 2a\,H_{ij}{}^{\mu\rho}\right) = \sigma^\mu{}_{ij}$$

The interaction with matter <u>different</u> from GR!

Canonical structure

Lagrangian variables: $\quad q^A = (b^i{}_\mu, A^{ij}{}_\mu, \lambda_{ij}{}^{\mu\nu})$

momenta: $\quad \pi_A = \dfrac{\partial \tilde{\mathcal{L}}}{\partial \dot{q}^A} \quad = (\pi_i{}^\mu, \pi_{ij}{}^\mu, \pi^{ij}{}_{\mu\nu})$

$\tilde{\mathcal{L}}$ given in Eq. (1)

- primary constraints:

$$\pi_i{}^0 \approx 0 \qquad\qquad \pi_{ij}{}^0 \approx 0$$
$$\pi_{ij}{}^{\mu\nu} \approx 0 \qquad\qquad \pi_{ij}{}^\alpha - 4\lambda_{ij}{}^{0\alpha} \approx 0 \tag{7}$$

these are sure constraints, ϕ_A.
the definition of $\pi_i{}^\alpha$ may lead to additional, extra constraints if A, B, C have specific values (if constraints).
Extra constraints – not important for our analysis.

- Hamiltonian

$$\mathcal{H}_c = N\mathcal{H}_\perp + N^\alpha \mathcal{H}_\alpha - \tfrac{1}{2}A^{ij}{}_0 \mathcal{H}_{ij} - \lambda_{ij}{}^{\alpha\beta} R^{ij}{}_{\alpha\beta} + \partial_\alpha D^\alpha$$

$N = n_k b^k{}_0\,, \qquad N^\alpha = h_{\bar{k}}{}^\alpha b^k{}_0$

lapse and shift functions

$$\mathcal{H}_T = \mathcal{H}_c + \underbrace{u^A \phi_A}_{\text{sure}} + \underbrace{(u\cdot\phi)}_{\text{extra}}$$

Consistency requirement on primary constraints,

$$\dot{\phi}_A \equiv \{\phi_A, H_T\} \approx 0$$

lead to

- secondary constraints, and/or
- determination of some multipliers.

Further consistency requirements produce nothing new.

The final result:

$$\mathcal{H}_T = \hat{\mathcal{H}}_T + \partial_\alpha \bar{D}^\alpha$$

$$\hat{\mathcal{H}}_T = \bar{\mathcal{H}}_c + u^i{}_0 \pi_i{}^0 + \tfrac{1}{2} u^{ij}{}_0 \pi_{ij}{}^0 + \tfrac{1}{4} u_{ij}{}^{\alpha\beta} \pi^{ij}{}_{\alpha\beta} + (u\cdot\phi)$$

$$\bar{\mathcal{H}}_c = N\bar{\mathcal{H}}_\perp + N^\alpha \bar{\mathcal{H}}_\alpha - \tfrac{1}{2} A^{ij}{}_0 \bar{\mathcal{H}}_{ij} - \lambda_{ij}{}^{\alpha\beta} \bar{\mathcal{H}}^{ij}{}_{\alpha\beta}$$

$\bar{\mathcal{H}}_\perp$, $\bar{\mathcal{H}}_\alpha$, $\bar{\mathcal{H}}_{ij}$, $\bar{\mathcal{H}}^{ij}{}_{\alpha\beta}$ known expressions.

	first class	second class
primary	$\pi_i{}^0$, $\pi_{ij}{}^0$, $\pi^{ij}{}_{\alpha\beta}$	$\phi_{ij}{}^\alpha$, $\pi^{ij}{}_{\alpha\beta}$
secondary	$\bar{\mathcal{H}}_\perp$, $\bar{\mathcal{H}}_\alpha$, $\bar{\mathcal{H}}_{ij}$, $\bar{\mathcal{H}}^{ij}{}_{\alpha\beta}$	

First class constraints are responsible for gauge symmetries:

$\pi_i{}^0$, $\pi_{ij}{}^0$ $\rightarrow$ local Poincare

$\pi^{ij}{}_{\alpha\beta}$ $\rightarrow$ local lambda.

The λ symmetry

If gauge transformations are given in terms of $\varepsilon(t)$, $\dot{\varepsilon}(t)$, then the gauge generator has the form

$$G = \varepsilon(t)\, G^{(0)} + \dot{\varepsilon}(t)\, G^{(1)}$$

where $G^{(0)}$, $G^{(1)}$ are defined by the conditions

$$G^{(1)} = C_{PFC} \leftarrow \text{primary first class}$$

$$G^{(0)} + \{G^{(1)}, H_T\} = C_{PFC}$$

$$\{G^{(0)}, H_T\} = C_{PFC}$$

(Castellani '81)

The only PFC acting on λ is $\pi^{ij}{}_{\alpha\beta}$.

Start with $G^{(1)} \to \pi^{ij}{}_{\alpha\beta} \Rightarrow$

$$G_A = \frac{1}{4}\dot{\varepsilon}_{ij}{}^{\alpha\beta}\,\pi^{ij}{}_{\alpha\beta} + \frac{1}{4}\varepsilon_{ij}{}^{\alpha\beta}\,S^{ij}{}_{\alpha\beta}$$

$S^{ij}{}_{\alpha\beta}$ found with the help of Castellani's procedure:

$$S^{ij}{}_{\alpha\beta} = -4\,\bar{\mathcal{H}}^{ij}{}_{\alpha\beta} + 2A^{[i}{}_{s0}\,\pi^{j]s}{}_{\alpha\beta}.$$

$\Rightarrow$

$$\delta_0^A \lambda_{ij}{}^{0\alpha} = \{\lambda_{ij}{}^{\alpha\beta}, G_A\} = \nabla_\beta\, \varepsilon_{ij}{}^{\alpha\beta}$$

$$\delta_0 \lambda_{ij}{}^{\alpha\beta} = \nabla_0\, \varepsilon_{ij}{}^{\alpha\beta}$$

These are <u>the only</u> non-trivial gauge transformations.

Where is the transformation $\underline{\delta_0^B \lambda_{ij}{}^{\alpha\beta} = \nabla_\gamma \varepsilon_{ij}{}^{\alpha\beta\gamma}}$, found in the Lagrangian formalism?

Define

$$\Pi^{ij}{}_{\alpha\beta\gamma} = \nabla_\alpha \pi^{ij}{}_{\beta\gamma} + \text{cyclic}(\alpha,\beta,\gamma) \qquad (11)$$

this is a linear combination of $\pi^{ij}{}_{\alpha\beta}$
hence $\Pi^{ij}{}_{\alpha\beta\gamma} = C_{PFC}$.
The related generator G_B will not be trully independent of the expression for G_A.

Since $\{\Pi^{ij}{}_{\alpha\beta\gamma}, H_T\} = 0 \Rightarrow$

$$G_B = -\tfrac{1}{4} \varepsilon_{ij}{}^{\alpha\beta\gamma} \nabla_\alpha \pi^{ij}{}_{\beta\gamma}$$

$$\underline{\delta_0^B \lambda_{ij}{}^{\alpha\beta} = \nabla_\gamma \varepsilon_{ij}{}^{\alpha\beta\gamma}}$$

this is the missing transf. found in the Lagr. formalism.

$\delta_0^B \lambda_{ij}{}^{\alpha\beta}$ are <u>not independent</u> gauge transf.
$\Rightarrow$ 6 parameters $\underline{\varepsilon_{ij}{}^{\alpha\beta\gamma}}$ in the Lagr. λ transformation can be completly <u>discarded</u>.

No. of independent λ gauge transf. = $\underline{18}$
(the number of $\varepsilon_{ij}{}^{\alpha\beta}$).

Hence, the 2nd field equation can completly determine $\lambda_{ij}{}^{\mu\nu}$
(No. of indep. eqs. = 18 = No. of indep. λ's)

③ TOPOLOGICAL TELEPARALLEL 3D GRAVITY

3-dimensional PGT

M_3, the isometry group $= P(1,2)$

generators $\quad P_i, \quad M_{ij} = -\epsilon^{ij}{}_k J^k$

gauge fields $\quad b^i{}_\mu \quad A^{ij}{}_\mu = -\epsilon^{ij}{}_k \omega^k{}_\mu$

field strengths: $T^i{}_{\mu\nu}, \quad R^{ij}{}_{\mu\nu} = -\epsilon^{ij}{}_k R^k{}_{\mu\nu}$

$$T^i{}_{\mu\nu} = \partial_\mu b^i{}_\nu + \epsilon^{imn}\omega_{m\mu} b_{n\nu} - (\mu \leftrightarrow \nu)$$

$$R^i{}_{\mu\nu} = \partial_\mu \omega^i{}_\nu - \partial_\nu \omega^i{}_\mu + \epsilon^{imn}\omega_{m\mu}\omega_{n\nu}$$

<u>GR</u> $\qquad I_0 = -\int d^3x\, bR = \int d^3x\, \epsilon^{\mu\nu\rho} b^i{}_\mu R_{i\nu\rho}$

$\delta b^i{}_\mu : \qquad \epsilon^{\mu\nu\rho} R_{i\nu\rho} = 0$

$\delta \omega^i{}_\mu : \qquad \epsilon^{\mu\nu\rho} T_{i\nu\rho} = 0 \quad \Rightarrow \quad$ Riemannian connection

Standard P(1,2) gauge theory

Lie algebra $\qquad [J_i, J_j] = \epsilon_{ijk} J^k, \quad [P_i, P_j] = 0$

$\qquad [J_i, P_j] = \epsilon_{ijk} P^k$

$A_\mu = e^i{}_\mu P_i + \omega^i{}_\mu J_i \qquad$ gauge field

$u = \xi^i P_i + \tau^i J_i \qquad$ gauge parameter

$$\delta_0 A_\mu = -\nabla_\mu u = -\partial_\mu u - [A_\mu, u]$$

$$\delta_0 b^i{}_\mu = -\nabla_\mu \xi^i - \epsilon^{ijk} b_{j\mu}\tau_k$$

$$\delta_0 \omega^i{}_\mu = -\nabla_\mu \tau^i$$

$\tau^i \to \tau^i - \xi^\rho \omega^i{}_\rho \Rightarrow$

$$\left.\begin{aligned} \delta_0 b^i{}_\mu &= \delta_0^{PGT} b^i{}_\mu - \xi^\rho \underline{T^i{}_{\mu\rho}} \\ \delta_0 \omega^i{}_\mu &= \delta_0^{PGT} \omega^i{}_\mu - \xi^\rho \underline{R^i{}_{\mu\rho}} \end{aligned}\right\} \text{equivalent to PGT, } \underline{\text{on shell}}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$
$$= P_i T^i{}_{\mu\nu} + J_i R^i{}_{\mu\nu} \,.$$

GR with cosm. constant: GR_Λ

$$I_\Lambda = -\int d^3x\,(bR + 2\Lambda)$$
$$= \int d^3x\, \epsilon^{\mu\nu\rho}\left(b^i{}_\mu R_{i\nu\rho} - \frac{\Lambda}{3}\epsilon_{ijk} b^i{}_\mu b^j{}_\nu b^k{}_\rho\right)$$

field eqs:

$$\epsilon^{\mu\nu\rho}\left(R_{i\nu\rho} - \Lambda \epsilon_{ijk} b^j{}_\nu b^k{}_\rho\right) = 0$$
$$\epsilon^{\mu\nu\rho} T_{i\nu\rho} = 0$$

$\Rightarrow$ const. curvature, Riemannian connect.

For $\Lambda = -1/\ell^2 < 0 \Rightarrow$ <u>AdS geometry</u>:

$$ds^2 = (1 + r^2/\ell^2)\,dt^2 - \frac{dr^2}{1 + r^2/\ell^2} - r^2 d\varphi^2$$

Chern-Simons formulation

$SL(2,R)$ CS theory: $\qquad A^i = A^i{}_\mu dx^\mu$

$$I_{CS} = \underline{k} \int_M \left(A^i dA^j + \tfrac{1}{3} \epsilon_{mn}{}^i A^m A^n A^j \right) \eta_{ij} \qquad (4)$$

Cartan's metric

$$\delta I_{CS} = \int_M \delta A^i F^j \eta_{ij} + \underbrace{\int_{\partial M} A^i \delta A^j \eta_{ij}}_{=0 \text{ for } A_+ = 0 \text{ on } \partial M}$$

With such <u>boundary conditions</u>, diff on ∂M defined by

$$\varepsilon^i = \xi^\mu A^i{}_\mu \quad \text{at } \partial M$$

reduce to <u>conformal transformations</u>:

$$\{ L_n, L_m \} = -i(n-m) L_{n+m} - \frac{c_0}{12} i n^3 \delta_{n+m,0}$$

Virasoro algebra with class. central charge
$c_0/12 = -4\pi \underline{k} \alpha$

Important <u>identity</u>: $\qquad A^i = \omega^i + \theta^i/\ell$
$(\theta^i = b^i{}_\mu dx^\mu)$ $\qquad \bar{A}^i = \omega^i - \theta^i/\ell$

$$I_\Lambda \Rightarrow \underline{\frac{1}{16\pi G} I_\Lambda = I_{CS}(A) - I_{CS}(\bar{A})}$$

$$\text{if } 4\pi k = \frac{\ell}{8G}$$

$\Rightarrow$ <u>In GR_Λ</u>, <u>we have conf. symmetry at ∂M</u>.

Topological 3d gravity with torsion

Conf. symm. at ∂M – is it possible in a theory of gravity with torsion ?

Topological or topological-like terms :

$$
\begin{aligned}
I_1 &= -\int d^3x\, b\,(aR + 2\Lambda) \\
I_2 &= \int \left(\omega^i d\omega_i + \tfrac{1}{3}\epsilon_{ijk}\,\omega^i\omega^j\omega^k\right) \qquad (6)\\
I_3 &= \int \theta^i T_i = \tfrac{1}{2}\int d^3x\, \epsilon^{\mu\nu\rho}\, b^i{}_\mu T_{i\nu\rho}
\end{aligned}
$$

General action: $\quad I = I_1 + \alpha_2 I_2 + \alpha_3 I_3 + I_M$

field eqs $\Rightarrow$

$$\underline{T_{ijk} = A\,\epsilon_{ijk}} \qquad A = \frac{\alpha_2\Lambda + \alpha_3 a}{\alpha_2\alpha_3 - a^2} \qquad (7a)$$

$$\underline{R_{ijk} = B\,\epsilon_{ijk}} \qquad B = -\frac{(\alpha_3)^2 + a\Lambda}{\alpha_2\alpha_3 - a^2} \qquad (7b)$$

$A = \Delta + K \quad$ + curvature identity $\Rightarrow$

$$\underline{R^{ij}{}_{\mu\nu}(\Delta) = -\Lambda_{eff}\,(b^i{}_\mu b^j{}_\nu - b^i{}_\nu b^j{}_\mu)} \qquad (8)$$

$$\Lambda_{eff} \equiv B - \tfrac{1}{4}A^2$$

a consequence of (7a, b)

- Witten's choice : $\alpha_2 = \alpha_3 = 0$
  $\Rightarrow$ <u>Riemannian</u> geometry , AdS.
- Our choice : $B = 0$
  $\Rightarrow$ <u>teleparallel</u> geometry , AdS

Teleparallel 3d gravity

$(\alpha_3)^2 + a\Lambda = 0$, $\qquad \alpha_2 = 0 \qquad \left[\quad \alpha_3 \equiv -a\cdot\frac{2}{\ell} \right]$

$\Downarrow$

$\underline{R^i{}_{\mu\nu} = 0}$ $\qquad$ $\uparrow$ for simplicity

$$\underline{I = -a\int d^3x\, b\left(R - \frac{8}{\ell^2}\right) - \frac{a}{\ell}\int d^3x\, \epsilon^{\mu\nu\rho} b^i{}_\mu T^i{}_{\nu\rho}}$$

field eqs : $\quad R^i{}_{\mu\nu} = 0$, $\quad T_{ijk} = \frac{2}{\ell}\epsilon_{ijk}$

$\Rightarrow$ a) <u>teleparallel</u> geometry

b) $\Lambda_{eff} = -\frac{1}{\ell^2} < 0 \Rightarrow$ <u>AdS</u> metric

<u>AdS solution</u> :

$$ds^2 = f^2 dt^2 - \frac{dr^2}{f^2} - r^2 d\varphi^2, \qquad f^2 = 1 + \frac{r^2}{\ell^2}$$

$\theta^0 = f\,dt \sim \frac{r}{\ell}dt$

$\theta^1 = \frac{1}{f}dr \sim \frac{\ell}{r}dt$

$\theta^2 = r\,d\varphi \sim r\,d\varphi$

$$b^i{}_\mu \sim \begin{pmatrix} \frac{r}{\ell} & 0 & 0 \\ 0 & \frac{\ell}{r} & 0 \\ 0 & 0 & r \end{pmatrix}$$

$$A = \Delta + K \quad \Rightarrow \quad \omega^i{}_\mu \sim \begin{pmatrix} \frac{r}{\ell^2} & 0 & -\frac{r}{\ell} \\ 0 & \frac{1}{r} & 0 \\ -\frac{r}{\ell^2} & 0 & \frac{r}{\ell} \end{pmatrix}$$

<u>Black hole solution</u>

$$ds^2 = N^2 dt^2 - N^{-2} dr^2 - r^2(N_\varphi dt + d\varphi)^2$$

$$N^2 = \left(-2M + \frac{r^2}{\ell^2} + \frac{J^2}{r^2}\right) \qquad N_\varphi = \frac{J}{r^2}$$

etc.

Asymptotic configuration

Asymptotically, the black hole metric

$$ds^2 \sim \frac{r^2}{\ell^2}dt^2 - \frac{\ell^2}{r^2}dr^2 - \underline{2J\,dt\,d\varphi} - r^2 d\varphi^2$$

coincides with the dominant contribution in the AdS metric, at least for $J=0$.

A metric $g_{\mu\nu}$ is said to be <u>assympt. AdS</u> if

a) $g_{\mu\nu} \sim$ AdS metric for $r \to \infty$

b) invariant under the AdS group $SO(2,2)$.

These conditions imply the following assymptotic behaviour for $g_{\mu\nu}$:

$$\bar{g}_{\mu\nu} = \begin{pmatrix} \frac{r^2}{\ell^2} + \mathcal{O}_0 & \mathcal{O}_3 & \mathcal{O}_0 \\ \mathcal{O}_3 & -\frac{\ell^2}{r^2} + \mathcal{O}_4 & \mathcal{O}_3 \\ \mathcal{O}_0 & \mathcal{O}_3 & -r^2 + \mathcal{O}_0 \end{pmatrix}$$

- leading terms are the same as in AdS case
- the $\mathcal{O}_n$ terms are chosen so as to ensure $SO(2,2)$ invariance.

$$\bar{g}_{\mu\nu} = \bar{b}^i{}_\mu \bar{b}^j{}_\nu \eta_{ij} \longrightarrow \bar{b}^i{}_\mu \quad (\text{not uniquely})$$

$$A = \Delta + K \longrightarrow \bar{\omega}^i{}_\mu .$$

$$\bar{b}^i{}_\mu = \begin{pmatrix} \frac{r}{\ell} + \mathcal{O}_1 & \mathcal{O}_4 & \mathcal{O}_1 \\ \mathcal{O}_2 & \frac{\ell}{r} + \mathcal{O}_3 & \mathcal{O}_2 \\ \mathcal{O}_1 & \mathcal{O}_4 & r + \mathcal{O}_1 \end{pmatrix} = \underline{\text{leading}} + \bar{B}^i{}_\mu$$

$$\bar{\omega}^i{}_\mu = \begin{pmatrix} \frac{r}{\ell^2} + \mathcal{O}_1 & \mathcal{O}_2 & -\frac{r}{\ell^2} + \mathcal{O}_1 \\ \mathcal{O}_2 & \frac{1}{r} + \mathcal{O}_3 & \mathcal{O}_2 \\ -\frac{r}{\ell^2} + \mathcal{O}_1 & \mathcal{O}_2 & \frac{r}{\ell} + \mathcal{O}_1 \end{pmatrix} = \underline{\text{lead}}. + \bar{\Omega}^i{}_\mu$$

Asymptotic symmetries

Ass. symmetry = a subset of gauge transf. which transforms ass. configrations $\bar{b}^i{}_\mu, \bar{\omega}^i{}_\mu$ into themselves, i.e.

$$\delta_0 \bar{b}^i{}_\mu = \varepsilon^{ijk} \theta_j \bar{b}_{k\mu} - \partial_\mu \xi^\rho \bar{b}^i{}_\rho - \xi\cdot\partial \bar{b}^i{}_\mu = \delta_0 \bar{B}^i{}_\mu$$

$$\delta_0 \bar{\omega}^i{}_\mu = -\nabla_\mu \theta^i - \partial_\mu \xi^\rho \cdot \bar{\omega}^i{}_\rho - \xi\cdot\partial\, \bar{\omega}^i{}_\rho = \delta_0 \bar{\Omega}^i{}_\mu$$

Assymptotic symm's def. in this way are <u>not isometries</u>, since $\delta_0 \bar{b}^i{}_\mu, \delta_0 \bar{\omega}^i{}_\mu \neq 0$.

Acting on $\bar{b}^i{}_\mu, \bar{\omega}^i{}_\mu$, these transformations change the form of subleading terms.

The form of ass. symmetries will be found in 3 steps.

<u>Step 1</u> Multiply 1st eq. by $\bar{\sigma}^{i}{}_{\nu}$, symmetrize in (μ,ν) : 6 conditions

$$-\partial_\mu \xi^\rho \cdot \bar{g}_{\nu\rho} - \partial_\nu \xi^\rho \cdot \bar{g}_{\mu\rho} - \xi \cdot \partial\, \bar{g}_{\mu\nu} = \delta_0 \bar{G}_{\mu\nu}$$

Using $\quad \xi^\rho = \sum_n \xi^\rho_n r^{-n} \quad \Longrightarrow$

$$\xi^0 = \ell \left[T + \frac{1}{2} \frac{\partial^2 T}{\partial t^2} \frac{\ell^4}{r^2} \right] + \mathcal{O}_4$$

$$\xi^2 = S - \frac{1}{2} \frac{\partial^2 S}{\partial \varphi^2} \cdot \frac{\ell^2}{r^2} + \mathcal{O}_4$$

$$\xi^1 = -\ell \left(\frac{\partial T}{\partial t} \right) r + \mathcal{O}_1$$

where

$$\frac{\partial T}{\partial \varphi} = \ell \frac{\partial S}{\partial t}, \qquad \frac{\partial S}{\partial \varphi} = \ell \frac{\partial T}{\partial t}$$

Two functions $T(t,\varphi)$ and $S(t,\varphi)$ define <u>conformal symmetry</u> :

$$x^\pm = \frac{1}{\ell} t \pm \varphi, \qquad T^\pm = T \pm S, \qquad \partial_\pm T^\mp = 0 .$$

$$L = \xi^\mu \partial_\mu \Big|_{T^- = 0} \to \sum L_n e^{inx^+}$$

$$\bar{L} = \xi^\mu \partial_\mu \Big|_{T^+ = 0} \to \sum \bar{L}_n e^{inx^-}$$

L_n, $\bar{L}_n$ satisfy two commuting Virasoro algebras:

$$[L_n, L_m] = +i(n-m) L_{n+m}, \qquad [L_n, \bar{L}_m] = 0$$

$$[\bar{L}_n, \bar{L}_m] = -i(n-m) \bar{L}_{n+m}$$

<u>Step 2</u> The remaining 3 conditions in the 1st. eq. can be used to determine the Lorentz parameters θ_i :

$$\theta^0 = -\frac{\ell^2}{r}\partial_0\partial_2 T + \mathcal{O}_3$$

$$\theta^2 = \frac{\ell^3}{r}\partial_0^2 T + \mathcal{O}_3$$

$$\theta^1 = \partial_2 T + \mathcal{O}_2 .$$

<u>Step 3.</u> Going to the 2nd. eq. we find that it does not lead to any new restrictions.

$\Rightarrow$ <u>Assymptotic symmetry</u> corresponding to the chosen assympt. configuration $\bar{b}^i{}_\mu, \bar{\omega}^i{}_\mu$ in the <u>topological teleparallel theory</u> (9) is given by 2d <u>conformal symmetry</u>.

* * *

Basic references

mb@phy.bg.ac.yu

1. – M. Blagojević, Gravitation and gauge symmetries (IOP, Bristol, 2001)

2. – M. Blagojević and M. Vasilić, Class. Quant. Grav. 17 (2000) 3785; (hep-th/0006080)
   – Ref. 1.

3. – J. D. Brown and M. Henneaux, Comm. Math. Phys. 104 (1986) 207
   – E. Witten, Nucl. Phys. B311 (1988) 46
   – E. Mielke and P. Baekler, Phys. Lett. 156 (1991) 399

   – M.B. and M. Vasilić, unpublished

Concluding remarks

- In Rieman-Cartan geometry
  - both $\underline{\tau}$ and $\underline{\sigma}$ are sources of gravity
  - <u>PE</u> is compatible with torsion
- In <u>4d</u>, TP theory is <u>obervationally equivalent to GR</u>, but <u>interaction</u> with spinning matter is different
- In <u>3d</u>, topological TP theory has <u>conformal symmetry</u> at ∂M